Quantum Field Theory
An Introduction

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Preface

Quantum field theory describes the interaction between elementary particles. The relativistic version of the theory underlies high energy physics, particle physics and astrophysics. The nonrelativistic form is applied - for instance – to condensed matter up to quantum Hall fluids.

In this book the quantum field theory is only applied to electrons, positrons and photons. Although their interactions are already treated by quantum electrodynamics – the extremely successful prototype of modern quantum field theories – this lead-in is chosen because it is instructive and opens the access to other phenomena.

This publication is conceived as an introduction. The detailed developments and the numerous references to preceding places make it easier to follow. However, knowledge of the elements of quantum mechanics, relativistic mechanics and electrodynamics is a prerequisite. We use the units of the system SI (MKSA-system), no Einstein convention of summation over repeated indices and no natural units (with $\hbar = c = 1$).

As in our previous publication, Pfeifer, W. , 2004, we utilize the following symbols: operators are written in bold letters, three- dimensional vectors are marked with an arrow, two- or four-spinors or -vectors and quantities with more dimensions are underlined and the symbols for matrices are doubly underlined.

Contents

1 The lagrangian formulation of classical mechanics

The lagrangian and the hamiltonian principle are elegant tools to solve mechanical problems. They often are used in quantum field theory. A short introduction is given here.

1.1 The lagrangian principle

We begin with a system of n mass points with masses m .The coordinate of every point has k components (for the most part $k=3$). Therefore our system is described by $f=n\cdot k$ components $q_i(t),\ j=1,2,\cdots,f$. The multiplet which contains all the components is named $\underline{q}(t)=\{q_1(t),q_2(t),\cdots,q_f(t)\}$.

In non-relativistic mechanics we define the Lagrange function (Lagrangian) for simple situations

$$L(\underline{q},\underline{\dot{q}})=T(\underline{\dot{q}})-V(\underline{q}) \tag{1.1.1}$$

where $T(\underline{\dot{q}})$ is the kinetic energy of the system and $V(\underline{q})$ is its potential energy with the well-known expressions

$$T(\underline{\dot{q}})=\frac{m}{2}\sum_{i=1}^{f}\dot{q}_i^{\,2},\quad i.e.\frac{\partial T(\underline{\dot{q}})}{\partial \dot{q}_j}=m\dot{q}_j \tag{1.1.2}$$

$$\text{and}\quad -\frac{\partial V}{\partial q_j}=F_j, \tag{1.1.3}$$

where F_j is the *j*'th component of the force on the corresponding particle. The second Newtonian law reads

$$\frac{d(m\dot{q}_j)}{dt}=F_j. \tag{1.1.4}$$

Inserting $(1.1.2)$ and $(1.1.3)$ one obtains

$$\frac{d}{dt}\frac{\partial T(\underline{\dot{q}})}{\partial \dot{q}_j}=-\frac{\partial V}{\partial q_j}. \tag{1.1.5}$$

Because neither does T depend on q_j nor V on $\dot{q}_j$ the following equation is equivalent to $(1.1.5)$

$$\frac{d}{dt}\frac{\partial L(\underline{q},\underline{\dot{q}})}{\partial \dot{q}_j} = \frac{\partial L(\underline{q},\underline{\dot{q}})}{\partial q_j} \quad , j = 1,\cdots,f. \tag{1.1.6}$$

These expressions are the Euler-Lagrange equations. They are also valid for non-Canonical variables and - surprisingly - for relativistic mechanics.

For later applications we define the generalized momenta

$$p_j = \frac{\partial L}{\partial \dot{q}_j}. \tag{1.1.7}$$

1.2 Hamiltons principle

We start with the Lagrangian, $(1.1.1)$, and restrict our discussion to the case where all the $q_j(t)$ are independent (not correlated by given functions). The **action** S of a system is defined as the following time integral

$$S = \int_{t_0}^{t_e} L\left(\underline{q}(t),\underline{\dot{q}}(t)\right) dt \tag{1.2.1}$$

The **Hamilton principle** states that varying the coordinate functions $\underline{q}(t)$ the action S attains an extremum as soon as the functions $\underline{q}(t)$ have the true physical course satisfying the equations of motion. In order to show its consequences we consider a variation of $S, \delta S,$ to an arbitrary neighbouring path and demand

$$\delta S = 0 \tag{1.2.2}$$

I.e

$$0 = \delta S = \int_{t_0}^{t_e} dt \sum_{i=1}^{f} \left(\frac{\partial L}{\partial q_i(t)} \delta q_i(t) + \frac{\partial L}{\partial \dot{q}_i(t)} \delta \dot{q}_i(t) \right) \tag{1.2.3}$$

with the condition

$$\delta \underline{q}(t_0) = \delta \underline{q}(t_e) = 0 \text{ and the relation}$$
$$\delta \dot{q}_i = \frac{d}{dt} \delta q_i \; . \tag{1.2.4}$$

With the partial integration

$$\int_{t_o}^{t_e} dt \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i(t) = \frac{\partial L}{\partial \dot{q}_i} \delta q_i(t) \Big|_{t_0}^{t_e} - \int_{t_0}^{t_e} dt \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) \delta q_i(t) \tag{1.2.5}$$

we obtain

$$0 = \int_{t_0}^{t_e} dt \sum_{i=1}^{f} \left(\frac{\partial L}{\partial q_i(t)} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i(t)} \right) \delta q_i(t) \, . \tag{1.2.6}$$

Taking into account that the functions $\delta q_i(t)$ are arbitrary, every summand in $(1.2.6)$ must vanish, i.e.

$$\frac{\partial L}{\partial q_i(t)} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i(t)} = 0\,, \quad i = 1,2,\cdots.f. \tag{1.2.7}$$

We have obtained the Euler-Lagrange equations of motion, $(1.1.6)$. This result confirms the Hamilton principle $(1.2.2)$.

1.3 Continuous systems

Now we want to apply Hamiltons principle to a continuous system. We start with a flexible string containing the mass ρ per unit length, stretched by the constant force F between two fixed points at $x = 0$ and $x = l$, say, but subject to small transverse displacements in a plane. The function $\varphi(x,t)$ is the transverse displacement from equilibrium with $\varphi(0,t) = \varphi(l,t) = 0$. If φ increases by $d\varphi$ along dx, the corresponding element of the string has the length

$$\sqrt{dx^2 + d\varphi^2} = \sqrt{1 + \left(\frac{d\varphi}{dx}\right)^2}\,dx \approx \left(1 + \frac{1}{2}\left(\frac{d\varphi}{dx}\right)^2\right) \cdot dx\,, \tag{1.3.1}$$

where we presuppose $\frac{d\varphi}{dx} \ll 1$. The element of the string along dx is enlarged by about $\frac{1}{2}\left(\frac{d\varphi}{dx}\right)^2 dx$ and its potential energy is increased by $F \cdot \frac{1}{2}\left(\frac{d\varphi}{dx}\right)^2 dx$. The potential energy of the whole string amounts to

$$V = \frac{1}{2}F \cdot \int_0^l \left(\frac{d\varphi}{dx}\right)^2 dx \tag{1.3.2}$$

Making use of the one-dimensional density ρ the whole kinetic energy is written as follows

$$T = \frac{1}{2}\rho \cdot \int_0^l \left(\frac{d\varphi}{dt}\right)^2 dx\,. \tag{1.3.3}$$

As in $(1.1.1)$ the Lagrangian reads

$$L = T - V, \tag{1.3.4}$$

which we write using the Lagrange density $\mathcal{L}(x,t)$ like this

$$L = \int_0^l \mathcal{L}(x,t)dx\,. \tag{1.3.4a}$$

From $(1.3.2)$ up to $(1.3.4a)$ we obtain

$$\mathcal{L}(x,t) = \frac{1}{2}\rho\left(\frac{d\varphi(x,t)}{dt}\right)^2 - \frac{1}{2}F\left(\frac{d\varphi(x,t)}{dx}\right)^2 . \qquad (1.3.5)$$

Thus, the Lagrange density depends on $\frac{d\varphi(x,t)}{dt} \equiv \dot{\varphi}$ and $\frac{d\varphi(x,t)}{dx} \equiv \varphi'$. In special cases it depends also on $\varphi(x,t)$ itself.

The action of the system reads analogously to $(1.2.1)$

$$S = \int_0^l dx \int_{t_0}^{t_e} \mathcal{L}(\dot{\varphi},\varphi',\varphi)\, dt \qquad (1.3.6)$$

We perform a variation of $\varphi(x,t)$ with the conditions

$$\delta\varphi(x,t_0) = \delta\varphi(x,t_e) = 0 \quad \text{for all } x \text{ and} \qquad (1.3.7)$$

$$\delta\varphi(0,t) = \delta\varphi(l,t) = 0 \quad \text{for all } t. \qquad (1.3.8)$$

As in $(1.2.2)$ the Hamilton principle states

$$\delta S = 0 . \qquad (1.3.9)$$

Differentiating partially we obtain

$$0 = \int_0^l dx \int_{t_0}^{t_e} dt \left(\frac{\partial \mathcal{L}}{\partial \dot{\varphi}} \delta\dot{\varphi} + \frac{\partial \mathcal{L}}{\partial \varphi'} \delta\varphi' + \frac{\partial \mathcal{L}}{\partial \varphi} \delta\varphi \right) \qquad (1.3.10)$$

with $\quad \delta\dot{\varphi} = \frac{d}{dt}\delta\varphi , \quad \delta\varphi' = \frac{d}{dx}\delta\varphi .$

As in $(1.2.5)$ partial integration produces

$$\int_{t_0}^{t_e} dt \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} \delta\dot{\varphi} = -\int_{t_0}^{t_e} dt \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\varphi}}\right)\delta\varphi \quad \text{and} \quad \int_0^l dx \frac{\partial \mathcal{L}}{\partial \varphi'} \delta\varphi' = -\int_0^l dx \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial \varphi'}\right)\delta\varphi \qquad (1.3.11)$$

$$\text{i.e.} \quad 0 = \int_0^l dx \int_{t_0}^{t_e} dt \left(-\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\varphi}}\right) - \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial \varphi'}\right) + \frac{\partial \mathcal{L}}{\partial \varphi} \right)\delta\varphi . \qquad (1.3.12)$$

Since the virtual displacement function $\delta\varphi(x,t)$ is arbitrary the integrand in $(1.3.12)$ must vanish

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\varphi}}\right) + \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial \varphi'}\right) - \frac{\partial \mathcal{L}}{\partial \varphi} = 0 . \qquad (1.3.13)$$

This is the Euler-Lagrange equation of the field $\varphi(x,t)$. We verify $(1.3.13)$ by inserting $\mathcal{L}$ of the string, $(1.3.5)$, in $(1.3.13)$, which yields

$$\rho\frac{d^2\varphi}{dt^2}-F\frac{d^2\varphi}{dx^2}=0\,. \tag{1.3.14}$$

This is the well-known wave equation for small amplitudes φ of a string.

If φ exists in a N-dimensional space with coordinates $x_1,x_2,\cdots,x_N$ the equation $(1.3.13)$ has to be replaced by

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\varphi}}\right)+\sum_{n=1}^{N}\frac{d}{dx_n}\left(\frac{\partial\mathcal{L}}{\partial\left(\dfrac{d\varphi}{dx_n}\right)}\right)-\frac{\partial\mathcal{L}}{\partial\varphi}=0 \qquad .(1.3.15)$$

If φ is a vectorial field with K components $\varphi^{(1)}\left(x_1,\cdots,x_N,t\right),\cdots,\varphi^{(K)}\left(x_1,\cdots,x_N,t\right)$ we have K Euler-Lagrange equations of the form

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\varphi}^{(k)}}\right)+\sum_{n=1}^{N}\frac{d}{dx_n}\left(\frac{\partial\mathcal{L}}{\partial\left(\dfrac{d\varphi^{(k)}}{dx_n}\right)}\right)-\frac{\partial\mathcal{L}}{\partial\varphi^{(k)}}=0 \quad , \quad k=1,\cdots,K\,. \tag{1.3.16}$$

The Lagrange function $\mathcal{L}$ plays an important rôle in quantum field theories. In our lead-in example (string) we have constructed $\mathcal{L}$ following the conventional method for simple mechanical problems. However, general rules for generating the Lagrange density cannot be given. Effectively, one has to guess the function $\mathcal{L}$ and to apply the Euler-Lagrange equations to it. The resulting equation must agree with the known equation of motion of the problem worked on.

For instance, we claim that the quantum mechanical behaviour of a mass is characterized by the following Lagrange density

$$\mathcal{L}\left(\varphi,\nabla\varphi,\dot{\varphi}\right)=\mathrm{i}\hbar\varphi^*\frac{d\varphi}{dt}-\frac{\hbar^2}{2m}\sum_{n}\frac{d\varphi^*}{dx_n}\frac{d\varphi}{dx_n}-V\left(\vec{x},t\right)\varphi^*\varphi \tag{1.3.17}$$

The function φ and the conjugate complex φ^* can be treated as independent fields. For φ^* and $N=3$ the Euler-Lagrange equation $(1.3.16)$ reads

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\varphi}^*}\right)=-\sum_{n=1}^{3}\frac{d}{dx_n}\left(\frac{\partial\mathcal{L}}{\partial\left(\dfrac{d\varphi^*}{dx_n}\right)}\right)+\frac{\partial\mathcal{L}}{\partial\varphi^*} \tag{1.3.18}$$

We insert $(1.3.17)$ into $(1.3.18)$:

$$0=\frac{\hbar^2}{2m}\sum_{n=1}^{3}\frac{d^2\varphi}{dx_n{}^2}+\mathrm{i}\hbar\frac{d\varphi}{dt}-V(\vec{x},t)\varphi\,, \tag{1.3.19}$$

which is the well-known Schrödinger equation. Thus, the choice of $\mathcal{L}$ in $(1.3.17)$ is justified.

1.4 The energy-momentum tensor

Here we derive the energy density and the density of the linear momentum of a field, which we choose as scalar. The conservation of the total energy and momentum will result.

We consider a uniform, infinitesimal space-time displacement. Every t is increased by the constant amount δt and analogously x_k by the spatially constant amount δx_k $(k=1,2,3)$. Consequently the change of φ is

$$\delta\varphi=\frac{d\varphi}{dt}\delta t+\sum_k\frac{d\varphi}{dx_k}\delta x_k\,. \tag{1.4.1}$$

We investigate the corresponding change of the Lagrange density $\mathcal{L}\left(\varphi,\frac{d\varphi}{dt},\frac{d\varphi}{dx_i}(i=1,2,3)\right)$. Since $\mathcal{L}$ does not depend explicitly on t or x_i we have

$$\delta\mathcal{L}=\frac{\partial\mathcal{L}}{\partial\varphi}\delta\varphi+\frac{\partial\mathcal{L}}{\partial\left(\frac{d\varphi}{dt}\right)}\delta\left(\frac{d\varphi}{dt}\right)+\sum_i\frac{\partial\mathcal{L}}{\partial\left(\frac{d\varphi}{dx_i}\right)}\delta\left(\frac{d\varphi}{dx_i}\right). \tag{1.4.2}$$

We take the expression $\frac{\partial\mathcal{L}}{\partial\varphi}$ from $(1.3.15)$, and from $(1.3.10)$ we have

$$\delta\left(\frac{d\varphi}{dt}\right)=\frac{d}{dt}\delta\varphi,\quad \delta\left(\frac{d\varphi}{dx_i}\right)=\frac{d}{dx_i}\delta\varphi\,, \tag{1.4.3}$$

which we insert in $(1.4.2)$:

$$\delta\mathcal{L} = \left(\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \left(\frac{d\varphi}{dt} \right)} \right) \delta\varphi + \sum_{i=1}^{3} \left(\frac{d}{dx_i} \frac{\partial \mathcal{L}}{\partial \left(\frac{d\varphi}{dx_i} \right)} \right) \delta\varphi$$

$$+ \frac{\partial \mathcal{L}}{\partial \left(\frac{d\varphi}{dt} \right)} \frac{d}{dt} \delta\varphi + \sum_{i=1}^{3} \frac{\partial \mathcal{L}}{\partial \left(\frac{d\varphi}{dx_i} \right)} \frac{d}{dx_i} \delta\varphi \qquad (1.4.4)$$

$$= \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \left(\frac{d\varphi}{dt} \right)} \cdot \delta\varphi \right) + \sum_{i=1}^{3} \frac{d}{dx_i} \left(\frac{\partial \mathcal{L}}{\partial \left(\frac{d\varphi}{dx_i} \right)} \cdot \delta\varphi \right).$$

By means of $(1.4.1)$ we obtain

$$\delta\mathcal{L} \quad = \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \left(\frac{d\varphi}{dt} \right)} \cdot \frac{d\varphi}{dt} \right) \cdot \delta t + \frac{d}{dt} \sum_k \left(\frac{\partial \mathcal{L}}{\partial \left(\frac{d\varphi}{dt} \right)} \cdot \frac{d\varphi}{dx_k} \right) \delta x_k$$

$$+ \sum_i \frac{d}{dx_i} \left(\frac{\partial \mathcal{L}}{\partial \left(\frac{d\varphi}{dx_i} \right)} \cdot \frac{d\varphi}{dt} \right) \delta t + .\sum_i \frac{d}{dx_i} \sum_k \left(\frac{\partial \mathcal{L}}{\partial \left(\frac{d\varphi}{dx_i} \right)} \cdot \frac{d\varphi}{dx_k} \right) \delta x_k . \qquad (1.4.5)$$

On the other hand, obviously holds

$$\delta\mathcal{L} = \frac{d\mathcal{L}}{dt} \delta t + \sum_k \frac{d\mathcal{L}}{dx_k} \delta x_k \quad , \quad k = 1,2,3. \qquad (1.4.6)$$

Comparing $(1.4.5)$ with $(1.4.6)$ and taking into account that δt and the δx_k's are independent we obtain from the δt-terms

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \left(\frac{d\varphi}{dt} \right)} \cdot \frac{d\varphi}{dt} \right) + \sum_i \frac{d}{dx_i} \left(\frac{\partial \mathcal{L}}{\partial \left(\frac{d\varphi}{dx_i} \right)} \frac{d\varphi}{dt} \right) - \frac{d\mathcal{L}}{dt} = 0 \qquad (1.4.7)$$

With the definitions

$$\frac{\partial \mathcal{L}}{\partial \left(\frac{d\varphi}{dt} \right)} \cdot \frac{d\varphi}{dt} - \mathcal{L} = T_0^0 \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial \left(\frac{d\varphi}{dx_i} \right)} \cdot \frac{d\varphi}{dt} = T_0^i \quad , \quad \mathrm{i} = 1,2,3 \qquad (1.4.8)$$

it reads

$$0=\frac{dT_0^0}{dt}+\sum_i\frac{dT_0^i}{dx_i}=\frac{dT_0^0}{dt}+div\vec{T}_0 \quad \text{with } \vec{T}_0=\left(T_0^{(1)},T_0^{(2)},T_0^3\right) \qquad (1.4.9)$$

Integrating $(1.4.9)$ over all the space, using the divergence theorem of Gauss and considering that the field vanishes at large distances yields

$$0=\frac{d}{dt}\int d^3xT_0^0+\int d^3x\,div\vec{T}_0=\frac{d}{dt}\int d^3xT_0^0+\int_{\text{surface}}\vec{\mathrm{T}}\bullet d\vec{\mathrm{A}}=\frac{d}{dt}\int d^3xT_0^0 \qquad (1.4.10)$$

Therefore the quantity $\int d^3xT_0^0$ remains constant and we expect that it is the energy of the field. Consequently, T_0^0 which is named the 00-element of the energy-momentum tensor, is the energy density, which is also denoted by $\mathcal{H}$:

$$\mathcal{H}=\frac{\partial\mathcal{L}}{\partial\left(\frac{d\varphi}{dt}\right)}\cdot\frac{d\varphi}{dt}-\mathcal{L} \qquad (1.4.11)$$

It is also named Hamilton density.

By means of the Lagrange density $(1.3.5)$ and by $(1.4.11)$ we calculate the energy density of the string

$$\begin{aligned}\mathcal{H}&=\rho\left(\frac{d\varphi(x,t)}{dt}\right)^2-\frac{1}{2}\rho\left(\frac{d\varphi(x,t)}{dt}\right)^2+\frac{1}{2}F\left(\frac{d\varphi(x,t)}{dx}\right)^2\\&=\frac{1}{2}\rho\left(\frac{d\varphi(x,t)}{dt}\right)^2+\frac{1}{2}F\left(\frac{d\varphi(x,t)}{dx}\right)^2,\end{aligned} \qquad (1.4.12)$$

as we expect due to $(1.3.2)$ and $(1.3.3)$. This result confirms our interpretation of T_0^0 as energy density.

If φ is a vectorial field (see (1.3.16)) the energy density reads

$$\mathcal{H}=\sum_{k=1}^{K}\frac{\partial\mathcal{L}}{\partial\left(\frac{d\varphi_k}{dt}\right)}\frac{d\varphi_k}{dt}-\mathcal{L}\,. \qquad (1.4.13)$$

Usually the derivative of $\mathcal{L}$ given above is denoted like this

$$\frac{\partial\mathcal{L}}{\partial\left(\frac{d\varphi_k}{dt}\right)}\equiv\frac{\partial\mathcal{L}}{\partial\dot{\varphi}_k}=\pi_k \qquad (1.4.14)$$

and is named canonically conjugate field of $\varphi_k\left(\vec{x},t\right)$. For instance, the canonically conjugate of a quantum mechanical field φ reads using $(1.3.17)$

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \mathrm{i}\hbar\varphi^* \qquad (1.4.15)$$

Due to $(1.4.11)$ and $(1.4.14)$ the energy density (Hamilton density) is

$$\mathcal{H}(\vec{x},t) = \pi(\vec{x},t)\cdot\dot{\varphi}(\vec{x},t) - \mathcal{L}(\vec{x},t), \qquad (1.4.16)$$

We go on equating the expression $(1.4.5)$ with $(1.4.6)$. From the δx_k-term we obtain

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{d\varphi}{dt}\right)}\cdot\frac{d\varphi}{dx_k}\right) + \sum_i \frac{d}{dx_i}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{d\varphi}{dx_i}\right)}\cdot\frac{d\varphi}{dx_k}\right) - \frac{\partial \mathcal{L}}{\partial x_k} = 0. \qquad (1.4.17)$$

We define the following elements of the energy-momentum tensor

$$T_k^0 = \frac{\partial \mathcal{L}}{\partial\left(\frac{d\varphi}{dt}\right)}\cdot\frac{d\varphi}{dx_k} \quad \text{and} \quad T_k^i = \frac{\partial \mathcal{L}}{\partial\left(\frac{d\varphi}{dx_i}\right)}\cdot\frac{d\varphi}{dx_k} \qquad (1.4.18)$$

and integrate $(1.4.17)$ as follows

$$\frac{d}{dt}\int d^3x T_k^0 + \int d^3x \sum_i \frac{dT_k^i}{dx_i} - \int d^2x \int dx_k \frac{d\mathcal{L}}{dx_k} = 0. \qquad (1.4.19)$$

With the same argument as for $(1.4.10)$ and defining $\vec{T}_k = \left(T_k^{(1)}, T_k^{(2)}, T_k^{(3)}\right)$ we obtain

$$0 = \frac{d}{dt}\int d^3x T_k^0 + \int_{\text{surface}} d\vec{A}\cdot\vec{T}_k - \int d^2x \left(\mathcal{L}\Big|_{-\infty}^{+\infty}\right) = \frac{d}{dt}\int d^3x T_k^0. \qquad (1.4.20)$$

Therefore $\int d^3x T_k^0$ is a constant. It must be the k'th component of the linear momentum of the whole field:

$$P_k = \int d^3x \frac{\partial \mathcal{L}}{\partial\left(\frac{d\varphi}{dt}\right)}\cdot\frac{d\varphi}{dx_k}. \qquad (1.4.21)$$

1.5 The Hamilton formalism, Poisson brackets

We will formulate Poisson brackets of the field. They are the starting point for the field quantization.

Generally, as in $(1.3.5)$ and in $(1.3.17)$, the Lagrange density $\mathcal{L}$ simultaneously depends on the values of the fields $\varphi(\vec{x},t), \frac{d\varphi(\vec{x},t)}{dt}, \frac{d\varphi(\vec{x},t)}{dx_n}$ and of the corresponding complex conjugate quantities. Functions F with such a dependence on functions are named functionals of the field. It is customary to denote a functional dependence by square brackets

$$F(\vec{x},t)=F\left[\varphi(\vec{x},t),\dot{\varphi}(\vec{x},t),\frac{d\varphi(\vec{x},t)}{dx_n},\cdots\right]. \tag{1.5.1}$$

For the moment we take only one field function $\varphi(\vec{x},t)$ and define the variation of the functional $F\left[\varphi(\vec{x},t)\right]$

$$\delta F[\varphi]=F[\varphi+\delta\varphi]-F[\varphi]. \tag{1.5.2}$$

The variation $\delta\varphi$ depends on space and time. We look into δF in more detail. We divide up all the space into small cells with central position vectors $\vec{x}_k$ and volumes ΔV_k. The average value of the field in ΔV_k is φ_k. We form a special variation of F supposing that only in one cell (number k) the value φ_k is varied by $\delta\varphi_k$ through which the function F varies in the whole space depending on $\vec{x}$ and t by $\delta F_k(\vec{x},t)$. I.e.

$$\delta F_k(\vec{x},t)=\frac{\partial F(\vec{x},t)}{\partial\varphi_k}\delta\varphi_k . \tag{1.5.3}$$

In first order approximation the effect of all variations $\delta\varphi_k$ on the spatial and temporal function $F(\vec{x},t)$ add up simply to

$$\delta F(\vec{x},t)=\sum_k \delta F_k(\vec{x},t)=\sum_k \frac{\partial F(\vec{x},t)}{\partial\varphi_k}\delta\varphi_k . \tag{1.5.4}$$

In order to construct a spatial integral we formulate

$$\delta F(\vec{x},t)=\sum_k \lim_{\Delta V_k\to 0}\Delta V_k \frac{1}{\Delta V_k}\cdot\frac{\partial F(\vec{x},t)}{\partial\varphi_k}\delta\varphi_k . \tag{1.5.5}$$

Denoting

$$\lim_{\Delta V_k\to 0}\frac{\partial F(\vec{x},t)}{\Delta V_k\partial\varphi_k}\equiv\frac{\delta F(\vec{x},t)}{\delta\varphi(\vec{x}',t)} \quad \text{with } \vec{x}_k \to \vec{x}' \tag{1.5.6}$$

we write $(1.5.5)$ like this

$$\delta F(\vec{x},t)=\int d^3x'\frac{\delta F(\vec{x},t)}{\delta\varphi(\vec{x}',t)}\delta\varphi(\vec{x}',t). \tag{1.5.7}$$

We introduce the Poisson brackets of fields being based on the field function $\varphi(\vec{x}'',t)$ and its canonically conjugate field $\pi(\vec{x}'',t)$, $(1.4.14)$. Given two functions $F(\vec{x},t)$ and $G(\vec{x}',t)$ we define the Poisson bracket

$$\left\{F(\vec{x},t),G(\vec{x}',t)\right\}_{BP} = \int d^3x'' \left(\frac{\delta F(\vec{x},t)}{\delta\varphi(\vec{x}'',t)} \frac{\delta G(\vec{x}',t)}{\delta\pi(\vec{x}'',t)} - \frac{\delta F(\vec{x},t)}{\delta\pi(\vec{x}'',t)} \frac{\delta G(\vec{x}',t)}{\delta\varphi(\vec{x}'',t)} \right). \quad (1.5.8)$$

The following special case will be interesting

$$F(\vec{x},t) = \varphi(\vec{x},t), \quad G(\vec{x}',t) = \pi(\vec{x}',t). \quad (1.5.9)$$

For this case we insert the first function of $(1.5.9)$ in $(1.5.7)$ and obtain

$$\delta\varphi(\vec{x},t) = \int d^3x'' \frac{\delta\varphi(\vec{x},t)}{\delta\varphi(\vec{x}'',t)} \delta\varphi(\vec{x}'',t) \quad (1.5.10)$$

On the other hand, a variation of φ can be written as

$$\delta\varphi(\vec{x},t) = \int d^3x'' \delta^3(\vec{x}-\vec{x}'')\, \delta\varphi(\vec{x}'',t). \quad (1.5.11)$$

Comparing $(1.5.10)$ with $(1.5.11)$ yields

$$\frac{\delta\varphi(\vec{x},t)}{\delta\varphi(\vec{x}'',t)} = \delta^3(\vec{x}-\vec{x}'') \quad (1.5.12)$$

and analogously $$\frac{\delta\pi(\vec{x}',t)}{\delta\pi(\vec{x}'',t)} = \delta^3(\vec{x}'-\vec{x}''). \quad (1.5.13)$$

Moreover, $$\frac{\delta\varphi(\vec{x},t)}{\delta\pi(\vec{x}'',t)} = \frac{\delta\pi(\vec{x}',t)}{\delta\varphi(\vec{x}'',t)} = 0 \quad (1.5.14)$$

holds since φ and π are independent functionals. We form the mutual Poisson brackets of the fields φ and π using $(1.5.8)$, $(1.5.12)$ up to $(1.5.14)$

$$\begin{aligned} \left\{\varphi(\vec{x},t),\pi(\vec{x}',t)\right\}_{PB} &= \int d^3x'' \frac{\delta\varphi(\vec{x},t)}{\delta\varphi(\vec{x}'',t)} \frac{\delta\pi(\vec{x}',t)}{\delta\pi(\vec{x}'',t)} = 0 \\ &= \int d^3x'' \delta^3(\vec{x}-\vec{x}'')\, \delta^3(\vec{x}'-\vec{x}'') = \delta^3(\vec{x}'-\vec{x}) \end{aligned} \quad (1.5.15)$$

and because of $(1.5.14)$ we obtain

$$\left\{\varphi(\vec{x},t),\varphi(\vec{x}',t)\right\}_{PB} = \left\{\pi(\vec{x},t),\pi(\vec{x}',t)\right\}_{PB} = 0. \quad (1.5.16)$$

In the next section the fields will be quantized by replacing the Poisson brackets by commutation relations between corresponding operators.

2 Canonical quantization

2.1 Nonrelativistic quantum fields

We remind to the step which leads from classical mechanics to quantum mechanics i.e. to the **first quantization** . The momentum $\vec{p}$ of a particle is replaced by the operator $\vec{\boldsymbol{p}} = -\mathrm{i}\hbar\vec{\nabla}$ which contains the gradient operator:

$$\vec{p} \quad \to \quad \vec{\boldsymbol{p}} = -\mathrm{i}\hbar\vec{\nabla}\,. \tag{2.1.1}$$

The position vector $\vec{x}$ is understood as an operator:

$$\vec{x} \quad \to \quad \vec{\boldsymbol{x}}\,. \tag{2.1.2}$$

If the combined operator $\boldsymbol{p}_x\boldsymbol{x}$ acts on a function $f(x,t)$ the result reads

$$\begin{aligned}\boldsymbol{p}_x\boldsymbol{x}\, f(x,t) &= \boldsymbol{p}_x\left(\boldsymbol{x}\, f(x,t)\right) = -\mathrm{i}\hbar\frac{d}{dx}\left(x\, f(x,t)\right)\\ &= -\mathrm{i}\hbar f(x,t) - x\mathrm{i}\hbar\frac{d}{dx}f(x,t) = -\mathrm{i}\hbar f(x,t) + \boldsymbol{x}\boldsymbol{p}_x f(x,t)\\ \text{or}\ \ \mathrm{i}\hbar f(x,t) &= \left(\boldsymbol{x}\boldsymbol{p}_x - \boldsymbol{p}_x\boldsymbol{x}\right)f(x,t).\end{aligned} \tag{2.1.3}$$

I.e. for the commutator

$$\left[\boldsymbol{x},\boldsymbol{p}_x\right]_- \equiv \boldsymbol{x}\boldsymbol{p}_x - \boldsymbol{p}_x\boldsymbol{x} \tag{2.1.4}$$

we have

$$\left[\boldsymbol{x},\boldsymbol{p}_x\right]_- = \mathrm{i}\hbar\,. \tag{2.1.5}$$

Analogous relations hold for the y - and for the x -coordinate.

The **second quantization** is also based on commutation rules for operators. We start with the field function $\varphi(\vec{x},t)$ and its canonically conjugate field $\pi(\vec{x},t)$, $(1.4.14)$. On the analogy of $(2.1.1/2.1.2)$ and using $(1.4.15)$ these fields are replaced by operators as follows

$$\begin{aligned}\varphi(\vec{x},t) \quad &\to \quad \boldsymbol{\varphi}(\vec{x},t)\\ \pi(\vec{x},t) = \mathrm{i}\hbar\varphi^*(\vec{x},t) \quad &\to \quad \boldsymbol{\pi}(\vec{x},t) = \mathrm{i}\hbar\boldsymbol{\varphi}^\dagger(\vec{x},t).\end{aligned} \tag{2.1.6}$$

We see that the complex conjugate wave function φ^* is replaced by the hermitean adjoint field operator $\boldsymbol{\varphi}^\dagger(\vec{x},t)$. What do these operators act on? As we will see, $(2.2.26)$, a one-particle state reads

$$\boldsymbol{\varphi}^\dagger(\vec{x},t)\left|0\right\rangle, \tag{2.1.7}$$

where $|0\rangle$ is the empty state named vacuum state, and a state of n identical particles is written as

$$C_n \boldsymbol{\varphi}^\dagger(\vec{x}_1,t)\cdots\cdot\boldsymbol{\varphi}^\dagger(\vec{x}_n,t)|0\rangle \qquad (2.1.8)$$

with the normalization factor C_n. Similarly to $(2.1.5)$ one demands that the operators $\boldsymbol{\varphi}$ and $\boldsymbol{\pi}$ meet commutator rules according to the Poisson bracket relation $(1.5.15)$ and $(1.5.16)$:

$$\{\varphi(\vec{x},t),\pi(\vec{x}\,',t)\}_{PB} = \delta^3(\vec{x}-\vec{x}\,') \quad \rightarrow \quad [\boldsymbol{\varphi}(\vec{x},t),\boldsymbol{\pi}(\vec{x}\,',t)]_- = \mathrm{i}\hbar\delta^3(\vec{x}-\vec{x}\,') \quad (2.1.9)$$

and analogously

$$[\boldsymbol{\varphi}(\vec{x},t),\boldsymbol{\varphi}(\vec{x}\,',t)]_- = [\boldsymbol{\pi}(\vec{x},t),\boldsymbol{\pi}(\vec{x}\,',t)]_- = 0\,. \qquad (2.1.10)$$

We stress that these relations hold for equal time.

Inserting $\boldsymbol{\pi}(\vec{x}\,',t)$ from $(2.1.6)$ into $(2.1.9/10)$ we have for nonrelativistic quantum mechanics

$$[\boldsymbol{\varphi}(\vec{x},t),\boldsymbol{\varphi}^\dagger(\vec{x}\,',t)]_- = \delta^3(\vec{x}-\vec{x}\,') \qquad (2.1.11)$$

$$\text{and} \qquad [\boldsymbol{\varphi}^\dagger(\vec{x},t),\boldsymbol{\varphi}^\dagger(\vec{x}\,',t)]_- = 0\,. \qquad (2.1.12)$$

We will see that the relations $(2.1.9)$ up to $(2.1.12)$ hold only for Bosons (with integer spin values). For Fermions (half integer spin values) the commutators $[...,...]_-$ have to be replaced by anticommutators $[...,...]_+$ defined by $[\boldsymbol{\alpha},\boldsymbol{\beta}]_+ = \boldsymbol{\alpha}\boldsymbol{\beta}+\boldsymbol{\beta}\boldsymbol{\alpha}$.

2.2 Quantization rules for Bose particles

We start with a quantum mechanical field function $\varphi(\vec{x},t)$ and expand it in terms of a complete set of orthonormalized wave functions $u_i(\vec{x})$:

$$\varphi(\vec{x},t) = \sum_i a_i(t)u_i(\vec{x}) \qquad (2.2.1)$$

as commonly practiced. For the second quantization, $(2.1.6)$, the expression $(2.2.1)$ is replaced as follows

$$\varphi(\vec{x},t) = \sum_i a_i(t)u_i(\vec{x}) \quad \rightarrow \quad \boldsymbol{\varphi}(\vec{x},t) = \sum_i \boldsymbol{a}_i(t)u_i(\vec{x}) \qquad (2.2.2)$$

Obviously the operator property of $\boldsymbol{\varphi}$ is now carried by the time-dependent expansion coefficient $\boldsymbol{a}_i(t)$, while the $u_i(\vec{x})$ are ordinary complex valued functions. The hermitean adjoint of $(2.2.2)$ reads

$$\boldsymbol{\varphi}^\dagger(\vec{x},t) = \sum_k \boldsymbol{a}_k^\dagger(t)\, u_k{}^*(\vec{x}) \tag{2.2.3}$$

As will be shown subsequent to $(2.2.20)$ the operators $\boldsymbol{a}_i$ and $\boldsymbol{a}_i^\dagger$ form together the particle number operator which ascertains the number of particles in the level (state)i. As mentioned above, the functions $u_k(\vec{x})$ form a complete and orthogonal system, i.e.

$$\int d^3x\, u_i{}^*(\vec{x})u_k(\vec{x}) = \delta_{\iota\kappa}, \tag{2.2.4}$$

$$\sum_k u_k(\vec{x})u_k{}^*(\vec{x}') = \delta^3(\vec{x}-\vec{x}'). \tag{2.2.5}$$

Inserting the expansions $(2.2.2)$ and $(2.2.3)$ into the equal time commutation relation $(2.1.11)$ leads to

$$\sum_{ik} u_i(\vec{x})u_k{}^*(\vec{x}')\left[\boldsymbol{a}_i(t),\boldsymbol{a}_k^\dagger(t)\right]_- = \delta^3(\vec{x}-\vec{x}'). \tag{2.2.6}$$

Apparently, if the commutator in $(2.2.6)$ vanishes for $i \neq k$ and if it equals 1 for $i = k$, the equation $(2.2.6)$ agrees with $(2.2.5)$. In this way we obtain the following commutation relation for the expansion coefficients (operators) $\boldsymbol{a}_i(t)$:

$$\left[\boldsymbol{a}_i(t,)\boldsymbol{a}_k^\dagger(t)\right]_- = \delta_{\iota\kappa}. \tag{2.2.7}$$

Analogously we have

$$\left[\boldsymbol{a}_i(t),\boldsymbol{a}_k(t)\right]_- = \left[\boldsymbol{a}_i^\dagger(t),\boldsymbol{a}_k^\dagger(t)\right]_- = 0. \tag{2.2.8}$$

Now we deal with the energy operator and the number operator in canonically quantized fields. The energy expression H for a field is calculated by means of the Hamilton density $\mathcal{H}$, $(1.4.11)$:

$$H = \int d^3x\, \mathcal{H}(\vec{x}) = \int d^3x \left(\frac{\partial \mathcal{L}}{\partial\left(\frac{d\varphi}{dt}\right)} \frac{d\varphi}{dt} - \mathcal{L} \right) \tag{2.2.9}$$

Taking $\mathcal{L}$ from $(1.3.17)$ (in quantum mechanics, with a constant potential $V(\vec{x})$) results in

$$H = \int d^3x \left(\mathrm{i}\hbar\varphi^* \frac{d\varphi}{dt} - \mathrm{i}\hbar\varphi^* \frac{d\varphi}{dt} + \frac{\hbar^2}{2m}\sum_{n=1}^{3} \frac{d\varphi^*}{dx_n}\frac{d\varphi}{dx_n} + V(\vec{x})\varphi^*\varphi \right), \tag{2.2.10}$$

which, after integration by parts with the usual asymptotic behaviour of φ, leads to the Hamiltonian

$$H = \int d^3x\, \varphi^*(\vec{x},t)\left(-\frac{\hbar^2}{2m}\sum_{n=1}^{3}\frac{d^2}{dx_n^2} + V(\vec{x})\right)\varphi(\vec{x},t). \qquad (2.2.11)$$

Following $(2.1.6)$ we replace φ and φ^* by operators and obtain the Hamilton operator $\boldsymbol{H}$ of the field of Schrödinger particles. At the same time we insert the expansions $(2.2.2)$ and $(2.2.3)$

$$\boldsymbol{H} = \int d^3x \sum_{ik} u_k{}^*(\vec{x})\left(-\frac{\hbar^2}{2m}\sum_{n=1}^{3}\frac{d^2}{dx_n^2} + V(\vec{x})\right)u_i(\vec{x})\boldsymbol{a}_k^\dagger(\boldsymbol{t})\,\boldsymbol{a}_i(t). \qquad (2.2.12)$$

Presupposing that $u_i(\vec{x})$ is an eigenfunction of the Hamiltonian $\left(-\frac{\hbar^2}{2m}\sum_{n=1}^{3}\frac{d^2}{dx_n^2} + V(\vec{x})\right)$ with the eigenvalue E_i and making use of the orthonormality of the u_i's, $(2.2.4)$, we obtain

$$\boldsymbol{H} = \sum_i \boldsymbol{a}_i^\dagger(t)\,\boldsymbol{a}_i(t)E_i\,. \qquad (2.2.13)$$

We now show that the time dependence of the right hand side disappears. The well-known Heisenberg equation of motion is also valid for field operators that is

$$\mathrm{i}\hbar\dot{\boldsymbol{a}}_j(t) = \left[\boldsymbol{a}_j, \boldsymbol{H}\right]_- \,. \qquad (2.2.14)$$

In the commutator we insert $(2.2.13)$ and use $(2.2.7)$ and $(2.2.8)$ like this

$$\begin{aligned}
\left[\boldsymbol{a}_j, \boldsymbol{H}\right]_- &= \sum_i \left[\boldsymbol{a}_j(t,), \boldsymbol{a}_i^\dagger(t)\boldsymbol{a}_i(t)\right]_- E_i \\
&= \sum_i \left(\boldsymbol{a}_j(t,)\boldsymbol{a}_i^\dagger(t)\boldsymbol{a}_i(t) - \boldsymbol{a}_i^\dagger(t)\boldsymbol{a}_i(t)\boldsymbol{a}_j(t,)\right)E_i \\
&= \sum_i \left(\boldsymbol{a}_j(t,)\boldsymbol{a}_i^\dagger(t) - \boldsymbol{a}_i^\dagger(t)\boldsymbol{a}_j(t,)\right)\boldsymbol{a}_i(t)E_i \\
&= \sum_i \delta_{ji}\boldsymbol{a}_i(t)\,E_i = \boldsymbol{a}_j(t)E_j
\end{aligned} \qquad (2.2.15)$$

Equations $(2.2.14)$ and $(2.2.15)$ say

$$\mathrm{i}\hbar\dot{\boldsymbol{a}}_j(t) = \boldsymbol{a}_j(t)E_j\,, \qquad (2.2.16)$$

which is solved by $\boldsymbol{a}_j(t) = e^{-\mathrm{i}E_j t/\hbar}\boldsymbol{a}_j(0) \equiv e^{-\mathrm{i}E_j t/\hbar}\boldsymbol{a}_j\,.$ (2.2.17)

The hermitean adjoint reads

$$\boldsymbol{a}_j^\dagger(t) = e^{+\mathrm{i}E_j t/\hbar}\boldsymbol{a}_j^\dagger \qquad (2.2.18)$$

and $(2.2.13)$ results in $\boldsymbol{H} = \sum_i \boldsymbol{a}_i^\dagger\boldsymbol{a}_i E_i\,,$ (2.2.19) which is constant in time. The operator $\boldsymbol{H}$ is the energy operator. If it acts on a system of identical particles its eigenvalue must be the energy of the system. Therefore, due to $(2.2.19)$, the constant expression $\boldsymbol{a}_i^\dagger \boldsymbol{a}_i$ must be a particle number operator. Its eigenvalue n_i is the number of identical particles in state i. In this way the eigenvalue of $\boldsymbol{H}$ is $\sum_i n_i E_i$, i.e. the total energy of the collective state of identical particles as we expect. Consequently, the eigenvalue of the operator

$$\begin{aligned} &\boldsymbol{N} = \sum_i \boldsymbol{a}_i^\dagger \boldsymbol{a}_i \quad \text{has the value} \\ &n = \sum_i n_i \ . \end{aligned} \qquad (2.2.20)$$

$\boldsymbol{N}$ is the operator of the total particle number n. Or, if $\left|\Phi_n^B\right\rangle$ is a collective state with n Boson particles the eigenvalue equation

$$\boldsymbol{N}\left|\Phi_n^B\right\rangle = n\left|\Phi_n^B\right\rangle \quad \text{holds.} \qquad (2.2.21)$$

The application of the operator $\boldsymbol{a}_j^\dagger$ to $\left|\Phi_\mathrm{n}^\mathrm{B}\right\rangle$ produces a new state vector. Let us calculate the particle number of the state $\boldsymbol{a}_j^\dagger\left|\Phi_n^B\right\rangle$ using the commutation relations $(2.2.7)$ and $(2.2.8)$ and equation $(2.2.21)$

$$\begin{aligned} \boldsymbol{N}\,\boldsymbol{a}_j^\dagger\left|\Phi_n^B\right\rangle &= \sum_i \boldsymbol{a}_i^\dagger \boldsymbol{a}_i\;\boldsymbol{a}_j^\dagger\left|\Phi_n^B\right\rangle = \sum_i \boldsymbol{a}_i^\dagger\left(\delta_{ij} + \boldsymbol{a}_j^\dagger \boldsymbol{a}_i\right)\left|\Phi_n^B\right\rangle \\ &= \sum_i\left(\delta_{ij}\boldsymbol{a}_i^\dagger + \boldsymbol{a}_j^\dagger \boldsymbol{a}_i^\dagger \boldsymbol{a}_i\right)\left|\Phi_n^B\right\rangle = \boldsymbol{a}_j^\dagger\left(1+\boldsymbol{N}\right)\left|\Phi_n^B\right\rangle \\ &= (n+1)\boldsymbol{a}_j^\dagger\left|\Phi_n^B\right\rangle. \end{aligned} \qquad (2.2.22)$$

Similarly one finds $\quad \boldsymbol{N}\,\boldsymbol{a}_j\left|\Phi_n^B\right\rangle = (n-1)\boldsymbol{a}_j\left|\Phi_n^B\right\rangle. \qquad (2.2.23)$

Thus, the operators $\boldsymbol{a}_j^\dagger$ and $\boldsymbol{a}_j$ have the effect of increasing and decreasing the particle number n and therefore are named **creation** and **annihilation operators** respectively. Here we redefine the vacuum state $|0\rangle$ (see (2.1.7)). It is defined as a state which is destroyed by application of any annihilation operator, i.e.

$$\boldsymbol{a}_j\,|0\rangle = 0 \quad \text{for all } j\,. \qquad (2.2.24)$$

We go on studying states with identical particles and make use of the vacuum state $|0\rangle$. Due to $(2.2.22)$

$$\boldsymbol{N}\,\boldsymbol{a}_j^\dagger\,|0\rangle = (0+1)\boldsymbol{a}_j^\dagger\,|0\rangle \qquad (2.2.25)$$

holds. Furthermore, starting with $(2.2.3)$ we have

$$\boldsymbol{N}\,\boldsymbol{\varphi}^\dagger(\vec{x},t)|0\rangle = \boldsymbol{N}\left(\sum_i \boldsymbol{a}_i^\dagger(t)u_i{}^*(\vec{x})\right)|0\rangle = 1\cdot\sum_i \boldsymbol{a}_i^\dagger|0\rangle u_i{}^*(\vec{x}) \quad \text{or}$$
$$\boldsymbol{N}\,\boldsymbol{\varphi}^\dagger(\vec{x},t)|0\rangle = 1\cdot\boldsymbol{\varphi}^\dagger(\vec{x},t)|0\rangle. \tag{2.2.26}$$

I.e. this is a one-particle state. Now we look into the state $\boldsymbol{\varphi}^\dagger(\vec{x}_1,t)\boldsymbol{\varphi}^\dagger(\vec{x}_2,t)|0\rangle$

$$\boldsymbol{N}\boldsymbol{\varphi}^\dagger(\vec{x}_1,t)\boldsymbol{\varphi}^\dagger(\vec{x}_2,t)|0\rangle = \boldsymbol{N}\sum_{ik}\boldsymbol{a}_i^\dagger(t)\boldsymbol{a}_k^\dagger(t)|0\rangle u_i{}^*(\vec{x}_1)u_k{}^*(\vec{x}_2) \tag{2.2.27}$$

with $$\boldsymbol{N}\,\boldsymbol{a}_i^\dagger\boldsymbol{a}_k^\dagger|0\rangle = \sum_l \boldsymbol{a}_l^\dagger\boldsymbol{a}_l\boldsymbol{a}_i^\dagger\boldsymbol{a}_k^\dagger|0\rangle = \sum_l \boldsymbol{a}_l^\dagger\left(\delta_{li}+\boldsymbol{a}_i^\dagger\boldsymbol{a}_l\right)\boldsymbol{a}_k^\dagger|0\rangle$$
$$= \boldsymbol{a}_i^\dagger\boldsymbol{a}_k^\dagger|0\rangle + \sum_l \boldsymbol{a}_l^\dagger\boldsymbol{a}_i^\dagger\left(\delta_{lk}+\boldsymbol{a}_k^\dagger\boldsymbol{a}_l\right)|0\rangle = 2\boldsymbol{a}_i^\dagger\boldsymbol{a}_k^\dagger|0\rangle. \tag{2.2.28}$$

Therefore $$\boldsymbol{N}\boldsymbol{\varphi}^\dagger(\vec{x}_1,t)\boldsymbol{\varphi}^\dagger(\vec{x}_2,t)|0\rangle = 2\,\boldsymbol{\varphi}^\dagger(\vec{x}_1,t)\boldsymbol{\varphi}^\dagger(\vec{x}_2,t)|0\rangle. \tag{2.2.29}$$

I.e. this is a two-particle state.

Because of $(2.1.12)$ all operators $\boldsymbol{\varphi}^\dagger(\vec{x}_k t)$ in the n-particle state $(2.1.8)$ commute among each other, this state is symmetric under permutation of coordinates. This means that the Schrödinger field subject to the quantization condition $(2.1.6)$ describes indistinguishable particles that obey Bose-Einstein statistics.

2.3 Quantization rules for Fermi particles

As we have indicated at the end of section 2.1, for Fermi particles (with half integer spin values) the field quantization rules $(2.1.9)$ up to $(2.1.12)$ have to be modified. These commutation relations among the field operators must be replaced by anticommutation relations like this

$$\left[\boldsymbol{\varphi}(\vec{x},t),\boldsymbol{\varphi}^\dagger(\vec{x}',t)\right]_+ = \delta^3(\vec{x}-\vec{x}') \tag{2.3.1}$$

$$\left[\boldsymbol{\varphi}(\vec{x},t),\boldsymbol{\varphi}(\vec{x}',t)\right]_+ = \left[\boldsymbol{\varphi}^\dagger(\vec{x},t),\boldsymbol{\varphi}^\dagger(\vec{x}',t)\right]_+ = 0. \tag{2.3.2}$$

Anticommutators are defined as $\left[\boldsymbol{\alpha},\boldsymbol{\beta}\right]_+ = \boldsymbol{\alpha\beta}+\boldsymbol{\beta\alpha}$.

Let us first consider which consequences the condition $(2.3.2)$ has on the symmetry of the localized collective states $(2.1.8)$ comprising n identical particles

$$\left|\Phi_n^{\mathrm{F}}\right\rangle = C_n\boldsymbol{\varphi}^\dagger(\vec{x}_1,t)\boldsymbol{\varphi}^\dagger(\vec{x}_2,t)\cdots\cdot\boldsymbol{\varphi}^\dagger(\vec{x}_n,t)|0\rangle \tag{2.3.3}$$

According to $(2.3.2)$ every transposition of neighbouring operators $\boldsymbol{\varphi}^\dagger$ in $(2.3.3)$ changes the sign of the whole state. Therefore this collective state is completely antisymmetric which is characteristic for Fermi particles.

In the same way as in $(2.2.2)$ the operator $\boldsymbol{\varphi}(\vec{x},t)$ can be expanded in a series with wave functions $u_i(\vec{x})$ and "coefficients" (operators) $\boldsymbol{a}_i(t)$. In analogy with $(2.2.4)$ up to $(2.2.6)$ it can be shown that

$$\left[\boldsymbol{a}_i(t),\boldsymbol{a}_k^\dagger(t)\right]_+ = \delta_{\iota\kappa} \tag{2.3.4}$$

$$\left[\boldsymbol{a}_i(t),\boldsymbol{a}_k(t)\right]_+ = \left[\boldsymbol{a}_i{}^\dagger(t),\boldsymbol{a}_k{}^\dagger(t)\right]_+ = 0\,. \tag{2.3.5}$$

For $i = k$ we have $$2(\boldsymbol{a}_i)^2 = 2(\boldsymbol{a}_i^\dagger)^2 = 0\,. \tag{2.3.6}$$

The Hamiltonian $(2.2.19)$ and the particle number operator $(2.2.20)$ hold still for Fermi particles.

The equation of motion $(2.2.14)$ is equally valid here and therefore the particular number operator

$$\boldsymbol{n}_i = \boldsymbol{a}_i^\dagger\boldsymbol{a}_i \tag{2.3.7}$$

of the level i is also constant in time (c.f. $(2.2.15)$ - $(2.2.20)$).

We look into the square of this operator taking into account $(2.3.4)$ and $(2.3.6)$

$$\boldsymbol{n}_i{}^2 = \boldsymbol{a}_i^\dagger\left(\boldsymbol{a}_i\boldsymbol{a}_i^\dagger\right)\boldsymbol{a}_i = \boldsymbol{a}_i^\dagger\left(1\text{-}\,\boldsymbol{a}_i^\dagger\boldsymbol{a}_i\right)\boldsymbol{a}_i = \boldsymbol{a}_i^\dagger\boldsymbol{a}_i = \boldsymbol{n}_i \tag{2.3.8}$$

The square of this operator $\boldsymbol{n}_i$ equals $\boldsymbol{n}_i$, and its eigenvalue n_i show the same behaviour

$$n_i{}^2 = n_i\,. \tag{2.3.9}$$

This equation has the solution 1 or 0, which agrees with the well-known fact that at best one Fermi particle can exist in a level.

We deal with a collective state $(2.3.3)$ with totally n Fermi particles and n_i particles in state i, $\left|\Phi^F_{n,n_i}\right\rangle$. It meets $\boldsymbol{n}_i\left|\Phi^F_{n,n_i}\right\rangle = n_i\left|\Phi^F_{n,n_i}\right\rangle$. Analogously to $(2.2.22)$ and $(2.2.23)$ using $(2.3.4)$ and $(2.3.6)$ we calculate

$$\begin{aligned}\boldsymbol{n}_i\,\boldsymbol{a}_i^\dagger\left|\Phi^F_{n,n_i=0}\right\rangle &= \boldsymbol{a}_i^\dagger\boldsymbol{a}_i\boldsymbol{a}_i^\dagger\left|\Phi^F_{n,n_i=0}\right\rangle = \boldsymbol{a}_i^\dagger\left(1-\boldsymbol{a}_i^\dagger\boldsymbol{a}\right)\left|\Phi^F_{n,n_i=0}\right\rangle \\ &= \left(\boldsymbol{a}_i^\dagger 1-0\boldsymbol{a}_i^\dagger\right)\left|\Phi^F_{n,n_i=0}\right\rangle = 1\cdot\boldsymbol{a}_i^\dagger\left|\Phi^F_{n,n_i=0}\right\rangle\end{aligned} \tag{2.3.10}$$

The eigenvalue 1 shows that the particle number of $\boldsymbol{a}_i^\dagger\left|\Phi^F_{n,n_i=0}\right\rangle$ is 1 in level i. Similarly we perform

$$\begin{aligned}\boldsymbol{n}_i\,\boldsymbol{a}_i\left|\Phi^F_{n,n_i=1}\right\rangle &= \boldsymbol{a}_i^\dagger\boldsymbol{a}_i\boldsymbol{a}_i\left|\Phi^F_{n,n_i=1}\right\rangle = \boldsymbol{a}_i^\dagger\cdot 0\left|\Phi^F_{n,n_i=1}\right\rangle = 0 \\ &= \text{eigenvalue of } \boldsymbol{n}_i\,.\end{aligned} \tag{2.3.11}$$

I.e. the operators $\boldsymbol{a}_i^\dagger$ and $\boldsymbol{a}_i$ are creation and annihilation operators respectively, as in section 2.2.

With the help of $(2.3.10),(2.3.11)$ and $(2.3.6)$ one finds

$$\begin{aligned} &\boldsymbol{a}_i^\dagger \left|\Phi^F_{n,n_i=1}\right\rangle = \left(\boldsymbol{a}_i^\dagger\right)^2 \left|\Phi^F_{n,n_i=0}\right\rangle = 0, \\ &\boldsymbol{a}_i \left|\Phi^F_{n,n_i=0}\right\rangle = \left(\boldsymbol{a}_i\right)^2 \left|\Phi^F_{n,n_i=1}\right\rangle = 0 \end{aligned} \qquad (2.3.12)$$

i.e. in a given level only 1 or 0 particle may exist as we have found above.

3 Spin-½ fields

3.1 The Dirac equation

The particles with spin ½ are described by the Dirac equation. For a free particle with the rest mass m it reads in conventional formulation (c.f. Pfeifer, 2004, p.11)

$$\mathrm{i}\hbar \frac{d\underline{\varphi}(\vec{x},t)}{dt} + \mathrm{i}\hbar c \sum_{i=1}^{3} \underline{\underline{\alpha}}_i \frac{d\underline{\varphi}(\vec{x},t)}{dx^{(i)}} - \underline{\underline{\beta}} mc^2 \underline{\varphi}(\vec{x},t) = 0 \qquad (3.1.1)$$

with $x^{(1)} = x, x^{(2)} = y, x^{(3)} = z$.

The Dirac states $\underline{\varphi}$ are column spinors and the $\underline{\underline{\alpha}}_i$'s and $\underline{\underline{\beta}}$ are 4×4 matrices. We choose the following representations

$$\underline{\underline{\alpha}}_1 = \begin{pmatrix} 0 & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & 0 \end{pmatrix}, \quad \underline{\underline{\alpha}}_2 = \begin{pmatrix} 0 & & & \text{-i} \\ & & \text{i} & \\ & \text{-i} & & \\ \text{i} & & & 0 \end{pmatrix}$$
$$\underline{\underline{\alpha}}_3 = \begin{pmatrix} 0 & & 1 & 0 \\ & & 0 & -1 \\ 1 & 0 & & \\ 0 & -1 & & 0 \end{pmatrix} \quad \underline{\underline{\beta}} = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}. \qquad (3.1.2)$$

Introducing the zeroth relativistic coordinate

$$x^0 = ct \qquad (3.1.2\text{b})$$

we obtain

$$\mathrm{i}\hbar c \frac{d\underline{\varphi}(\vec{x},t)}{dx^0} + \mathrm{i}\hbar c \sum_{i=1}^{3} \underline{\underline{\alpha}}_i \frac{d\underline{\varphi}(\vec{x},t)}{dx^{(i)}} - \underline{\underline{\beta}} mc^2 \underline{\varphi}(\vec{x},t) = 0 . \qquad (3.1.3)$$

We define

$$\underline{\underline{\alpha}}_0 = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}, \qquad (3.1.4)$$

i.e.

$$\mathrm{i}\hbar c \sum_{\nu=0}^{3} \underline{\underline{\alpha}}_\nu \frac{d\underline{\varphi}(\vec{x},t)}{dx^{(\nu)}} - \underline{\underline{\beta}} mc^2 \underline{\varphi}(\vec{x},t) = 0 \qquad (3.1.5)$$

The definition $(3.1.2\text{b})$ says $\frac{d\underline{\varphi}(\vec{x},t)}{dx^0} = \frac{\dot{\underline{\varphi}}(\vec{x},t)}{c}$ and $(3.1.3)$ reads also

$$\mathrm{i}\hbar \dot{\underline{\varphi}}(\vec{x},t) + \mathrm{i}\hbar c \sum_{i=1}^{3} \underline{\underline{\alpha}}_i \frac{d\underline{\varphi}(\vec{x},t)}{dx^{(i)}} - \underline{\underline{\beta}} mc^2 \underline{\varphi}(\vec{x},t) = 0 . \qquad (3.1.6)$$

The hermitean conjugate field $\underline{\varphi}^\dagger(\vec{x},t)$ is defined as a line spinor with the conjugate complex elements of $\underline{\varphi}(\vec{x},t)$. We will treat the spinors $\underline{\varphi}$ and $\underline{\varphi}^\dagger$ as independent fields, each having four components. We claim that the following Lagrange density $\mathcal{L}$ can be chosen

$$\mathcal{L} = \mathrm{i}\hbar\underline{\varphi}^\dagger\underline{\dot{\varphi}} + \mathrm{i}\hbar c\underline{\varphi}^\dagger\sum_{i=1}^{3}\underline{\underline{\alpha}}_i\frac{d\underline{\varphi}}{dx^{(i)}} - mc^2\underline{\varphi}^\dagger\underline{\underline{\beta}}\underline{\varphi}\,, \tag{3.1.7}$$

where $\underline{\varphi}^\dagger\underline{\varphi}$ means

$$\underline{\varphi}^\dagger\underline{\dot{\varphi}} = \sum_{k=1}^{4}\varphi_k^\dagger\dot{\varphi}_k\,. \tag{3.1.8}$$

We confirm this form by applying the Euler-Lagrange equation $(1.3.16)$. For the time being we use the derivations with regard to $\underline{\varphi}^\dagger$

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\varphi}_k^\dagger}\right) + \sum_{i=1}^{3}\frac{d}{dx^{(i)}}\frac{\partial\mathcal{L}}{\partial\left(\dfrac{d\varphi_k^\dagger}{dx^{(i)}}\right)} - \frac{\partial\mathcal{L}}{\partial\varphi_k^\dagger} = 0 \quad \text{for } k = 1,\cdots,4 \tag{3.1.9}$$

and obtain

$$0 + 0 - \mathrm{i}\hbar\dot{\varphi}_k - \mathrm{i}\hbar c\sum_{i=1}^{3}\underline{\underline{\alpha}}_i\frac{d\varphi_k}{dx^{(i)}} + mc^2\left(\underline{\underline{\beta}}\underline{\varphi}\right)_k = 0\,, \tag{3.1.10}$$

which is one component of the Dirac equation $(3.1.6)$. If we replace $\varphi_k^\dagger$ in $(3.1.9)$ by φ_k we get in the same way

$$\mathrm{i}\hbar\dot{\varphi}_k^\dagger + \mathrm{i}\hbar c\sum_{i=1}^{3}\left(\frac{d\varphi_k^\dagger}{dx^{(i)}}\underline{\underline{\alpha}}_i\right)_k + mc^2\left(\underline{\varphi}^\dagger\underline{\underline{\beta}}\right)_k = 0\,, \tag{3.1.11}$$

which is one component of the hermitean conjugate of $(3.1.6)$. Thus, the expression $(3.1.7)$ for the Lagrange density is justified.

Due to $(1.4.14)$ the canonically conjugate of φ_k is

$$\pi_k = \frac{\partial\mathcal{L}}{\partial\dot{\varphi}_k} \tag{3.1.12}$$

and from $(3.1.7)$ we obtain

$$\pi_k = \mathrm{i}\hbar\varphi_k^\dagger\,. \tag{3.1.13}$$

We calculate the Hamilton density $\mathcal{H}$ by means of $(1.4.13)$ with $K = 4$ for $\underline{\varphi}$ and also for $\underline{\varphi}^\dagger$

$$\mathcal{H} = \sum_{k=1}^{4}\frac{\partial\mathcal{L}}{\partial\left(\dfrac{d\varphi_k}{dt}\right)}\frac{d\varphi_k}{dt} + \sum_{k=1}^{4}\frac{\partial\mathcal{L}}{\partial\left(\dfrac{d\varphi_k^\dagger}{dt}\right)}\frac{d\varphi_k^\dagger}{dt} - \mathcal{L}\,. \tag{3.1.14}$$

We insert $\mathcal{L}$ from $(3.1.7)$ and obtain

$$\begin{aligned}
&\mathcal{H} \\
&= \mathrm{i}\hbar\sum_{k=1}^{4}\varphi_k^\dagger\dot{\varphi}_k + 0 - \mathrm{i}\hbar\sum_{k=1}^{4}\varphi_k^\dagger\dot{\varphi}_k - \mathrm{i}\hbar c\sum_{k=1}^{4}\sum_{i=1}^{3}\varphi_k^\dagger\left(\underline{\underline{\alpha}}_i\frac{d\underline{\varphi}}{dx^{(i)}}\right)_k + mc^2\sum_{k=1}^{4}\varphi_k^\dagger\left(\underline{\underline{\beta}}\underline{\varphi}\right)_k \quad (3.1.15) \\
&= -\mathrm{i}\hbar c\sum_{k=1}^{4}\sum_{i=1}^{3}\varphi_k^\dagger\left(\underline{\underline{\alpha}}_i\frac{d\underline{\varphi}}{dx^{(i)}}\right)_k + mc^2\sum_{k=1}^{4}\varphi_k^\dagger\left(\underline{\underline{\beta}}\underline{\varphi}\right)_k \quad .
\end{aligned}$$

We write down the four-dimensional current density vector $\underline{j}(\vec{x},t)$ of a spin-½ particle with charge e. In Pfeifer, 2004, p. 20, the spatial components are derived

$$j^{(i)}(\vec{x},t) = ec\underline{\varphi}^\dagger(\vec{x},t)\underline{\underline{\alpha}}_i\underline{\varphi}(\vec{x},t) = ec\sum_{k,l=1}^{4}\varphi_k^*(\vec{x},t)\left(\underline{\underline{\alpha}}_i\right)_{kl}\varphi_l(\vec{x},t),\ i = 1,2,3\,. \quad (3.1.16)$$

The zero'th component is the charge density

$$c\rho = j^0(\vec{x},t) = ec\underline{\varphi}^\dagger(\vec{x},t)\underline{\varphi}(\vec{x},t). \quad (3.1.17)$$

Therefore, the total charge reads

$$Q = \int d^3x\ j^0(\vec{x},t)/c = e\int d^3x\ \underline{\varphi}^\dagger(\vec{x},t)\underline{\varphi}(\vec{x},t). \quad (3.1.18)$$

The four-vector $\underline{j}(\vec{x},t)$ is relativisticly contravariant.

3.2 Wave functions of free Dirac particles

The following wave function solves the Dirac equation $(3.1.1)$ (c.f. Pfeifer, 2004, p.31/33)

$$\underline{\varphi}(\vec{x},t,\vec{p}) = \frac{N}{\sqrt{V}}\begin{pmatrix}\underline{\varphi}_0\\ \underline{\chi}_0\end{pmatrix}e^{\mathrm{i}\left(\vec{p}\vec{x}-E(\vec{p})t\right)/\hbar}, \quad (3.2.1)$$

where $$E(\vec{p}) = \sqrt{m^2c^4 + c^2\vec{p}^2};\quad \vec{p} = \left(p_x,p_y,p_z\right) = \left(p^{(1)},p^{(2)},p^{(3)}\right) \quad (3.2.2)$$

and $$N = \sqrt{\frac{E(\vec{p}) + mc^2}{2E(\vec{p})}}\,. \quad (3.2.3)$$

We have introduced a representative volume V. The ortho-normalized two-spinor $\underline{\varphi}_0$ is chosen as follows

$$\underline{\varphi}_0 = \begin{pmatrix}1\\0\end{pmatrix}\ \text{or}\ \begin{pmatrix}0\\1\end{pmatrix}\ \text{respectively.} \quad (3.2.4)$$

The spinor $\underline{\chi}_0$ is related to $\underline{\varphi}_0$ like this (Pfeifer, 2004, p.32)

$$\underline{\chi}_0 = \frac{c\sum_{i=1}^{3} p^{(i)}\underline{\underline{\sigma}}_i}{E(\vec{p})+mc^2}\underline{\varphi}_0 \tag{3.2.5}$$

or
$$\underline{\varphi}_0 = \frac{c\sum_{i=1}^{3} p^{(i)}\underline{\underline{\sigma}}_i}{E(\vec{p})-mc^2}\underline{\chi}_0 \quad \text{respectively.} \tag{3.2.6}$$

The expression $(3.2.6)$ is discarded because it is diverging for $p \to 0$. The Pauli spin matrices read

$$\underline{\underline{\sigma}}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \underline{\underline{\sigma}}_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \underline{\underline{\sigma}}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3.2.7}$$

and the expression $\sum_{i=1}^{3} p^{(i)}\underline{\underline{\sigma}}_i = \begin{pmatrix} p^{(3)} & p^{(1)}-\mathrm{i}p^{(2)} \\ p^{(1)}+\mathrm{i}p^{(2)} & -p^{(3)} \end{pmatrix}$ holds. (3.2.8)

With the help of $(3.2.3)$ and $(3.2.5)$ the expression $(3.2.1)$ yields the following wave functions for both variants $(3.2.4)$

$$\varphi_1(\vec{x},t,\vec{p}) = \frac{1}{\sqrt{V}}\sqrt{\frac{E(\vec{p})+mc^2}{2E(\vec{p})}}\begin{pmatrix} 1 \\ 0 \\ \dfrac{cp^{(3)}}{E(\vec{p})+mc^2} \\ \dfrac{c\left(p^{(1)}+\mathrm{i}p^{(2)}\right)}{E(\vec{p})+mc^2} \end{pmatrix} e^{\mathrm{i}\left(\vec{p}\vec{x}-E(\vec{p})t\right)/\hbar}, \tag{3.2.9}$$

$$\varphi_2(\vec{x},t,\vec{p}) = \frac{1}{\sqrt{V}}\sqrt{\frac{E(\vec{p})+mc^2}{2E(\vec{p})}}\begin{pmatrix} 0 \\ 1 \\ \dfrac{c\left(p^{(1)}-\mathrm{i}p^{(2)}\right)}{E(\vec{p})+mc^2} \\ \dfrac{-cp^{(3)}}{E(\vec{p})+mc^2} \end{pmatrix} e^{\mathrm{i}\left(\vec{p}\vec{x}-E(\vec{p})t\right)/\hbar}. \tag{3.2.10}$$

We look into the following similar function (with negative exponent)

$$\frac{N}{\sqrt{V}}\begin{pmatrix} \underline{\varphi}_0' \\ \underline{\chi}_0' \end{pmatrix} e^{-\mathrm{i}\left(\vec{p}\vec{x}-E(\vec{p})t\right)/\hbar}. \tag{3.2.11}$$

Its wave moves in the direction of $\vec{p}$. Because E is positive (c.f.$(3.2.2)$) the quantum mechanical energy operator $\mathrm{i}\hbar\dfrac{d}{dt}$ has a negative eigenvalue. That's why this function has been excluded in the framework of one-particle physics in Pfeifer, 2004. However, in quantum field theory this solution is used to develop the fields.

In the same way as for $(3.2.5)$ and $(3.2.6)$ the following relations between $\underline{\varphi}_0'$ and $\underline{\chi}_0'$ (see $(3.2.11)$) can be derived

$$\underline{\chi}_0{}' = \frac{c\sum_{i=1}^{3} p^{(i)} \underline{\underline{\sigma}}_i}{E(\vec{p}) - mc^2} \underline{\varphi}_0' \tag{3.2.12}$$

or

$$\underline{\varphi}_0' = \frac{c\sum_{i=1}^{3} p^{(i)} \underline{\underline{\sigma}}_i}{E(\vec{p}) + mc^2} \underline{\chi}_0' . \tag{3.2.13}$$

Only the second expression makes sense physically because it is not divergent for $p \to 0$. For $\underline{\chi}_0'$ we choose the ortho-normalized form

$$\underline{\chi}_0' = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ respectively.} \tag{3.2.14}$$

We insert $(3.2.13)$ in $(3.2.11)$ and obtain the following wave functions for both variants $(3.2.14)$

$$\varphi_3(\vec{x},t,\vec{p}) = \frac{1}{\sqrt{V}} \sqrt{\frac{E(\vec{p}) + mc^2}{2E(\vec{p})}} \begin{pmatrix} \dfrac{cp^{(3)}}{E(\vec{p}) + mc^2} \\ \dfrac{c\left(p^{(1)} + \mathrm{i}p^{(2)}\right)}{E(\vec{p}) + mc^2} \\ 1 \\ 0 \end{pmatrix} e^{-\mathrm{i}\left(\vec{p}\vec{x} - E(\vec{p})t\right)/\hbar} \tag{3.2.15}$$

$$\varphi_4(\vec{x},t,\vec{p}) = \frac{1}{\sqrt{V}} \sqrt{\frac{E(\vec{p}) + mc^2}{2E(\vec{p})}} \begin{pmatrix} \dfrac{c\left(p^{(1)} - \mathrm{i}p^{(2)}\right)}{E(\vec{p}) + mc^2} \\ \dfrac{-cp^{(3)}}{E(\vec{p}) + mc^2} \\ 0 \\ 1 \end{pmatrix} e^{-\mathrm{i}\left(\vec{p}\vec{x} - E(\vec{p})t\right)/\hbar} . \tag{3.2.16}$$

We introduce the sign parameter ε_r which denotes

$$\begin{aligned} \varepsilon_r &= 1 \quad \text{for } r = 1, 2 \\ \varepsilon_r &= -1 \text{ for } r = 3, 4. \end{aligned} \tag{3.2.17}$$

With its help the expressions $(3.2.9),(3.2.10),(3.2.15)$ and $(3.2.16)$ can be written as follows

$$\varphi_r\left(\vec{x},t,\vec{p}\right)=\frac{1}{\sqrt{V}}\sqrt{\frac{mc^2}{E\left(\vec{p}\right)}}\,\underline{w}_r\left(\vec{p}\right)e^{\mathrm{i}\varepsilon_r\left(\vec{p}\vec{x}-E\left(\vec{p}\right)t\right)/\hbar} \tag{3.2.18}$$

with

$$w_1\left(\vec{p}\right)=\sqrt{\frac{E\left(\vec{p}\right)+mc^2}{2mc^2}}\begin{pmatrix}1\\0\\\dfrac{cp^{(3)}}{E\left(\vec{p}\right)+mc^2}\\\dfrac{c\left(p^{(1)}+\mathrm{i}p^{(2)}\right)}{E\left(\vec{p}\right)+mc^2}\end{pmatrix}$$

$$w_2\left(\vec{p}\right)=\sqrt{\frac{E\left(\vec{p}\right)+mc^2}{2mc^2}}\begin{pmatrix}0\\1\\\dfrac{c\left(p^{(1)}\text{-}\mathrm{i}p^{(2)}\right)}{E\left(\vec{p}\right)+mc^2}\\\dfrac{-cp^{(3)}}{E\left(\vec{p}\right)+mc^2}\end{pmatrix}$$

$$w_3\left(\vec{p}\right)=\sqrt{\frac{E\left(\vec{p}\right)+mc^2}{2mc^2}}\begin{pmatrix}\dfrac{cp^{(3)}}{E\left(\vec{p}\right)+mc^2}\\\dfrac{c\left(p^{(1)}+\mathrm{i}p^{(2)}\right)}{E\left(\vec{p}\right)+mc^2}\\1\\0\end{pmatrix}$$

$$w_4\left(\vec{p}\right)=\sqrt{\frac{E\left(\vec{p}\right)+mc^2}{2mc^2}}\begin{pmatrix}\dfrac{c\left(p^{(1)}\text{-}\mathrm{i}p^{(2)}\right)}{E\left(\vec{p}\right)+mc^2}\\\dfrac{-cp^{(3)}}{E\left(\vec{p}\right)+mc^2}\\0\\1\end{pmatrix} \tag{3.2.19}$$

The functions $\varphi_r\left(\vec{x},t,\vec{p}\right)$ are normalized to 1 within the volume V (Pfeifer, 2004, p.33). This procedure is named box normalization. The line-spinor $\underline{w}_r^{\dagger}\left(\vec{p}\right)$ contains the conjugate complex elements of $\underline{w}_r\left(\vec{p}\right)$. We show that the following relation holds

$$\underline{w}_r^{\dagger}\left(\varepsilon_r\cdot\vec{p}\right)\underline{w}_{r'}\left(\varepsilon_{r'}\cdot\vec{p}\right)=\delta_{rr'}\frac{E\left(\vec{p}\right)}{mc^2}\,. \tag{3.2.20}$$

As a check, for $r = r' = 1$ we calculate the following product using $(3.2.19)$

$$\underline{w}_1^{\dagger}(\varepsilon_1\vec{p})\,\underline{w}_1(\varepsilon_1\vec{p}) = \frac{E(\vec{p})+mc^2}{2mc^2} \cdot \left(1, 0, \frac{cp^{[3]}}{E(\vec{p})+mc^2}, \frac{c\left(p^{(1)}-\mathrm{i}p^{(2)}\right)}{E(\vec{p})+mc^2}\right) \begin{pmatrix} 1 \\ 0 \\ \dfrac{cp^{(3)}}{E(\vec{p})+mc^2} \\ \dfrac{c\left(p^{(1)}+\mathrm{i}p^{(2)}\right)}{E(\vec{p})+mc^2} \end{pmatrix}$$

$$= \frac{E(\vec{p})+mc^2}{2mc^2}\left(1+\frac{c^2\left(p^{(1)2}+p^{(2)2}+p^{(3)2}\right)}{\left(E(\vec{p})+mc^2\right)^2}\right) \tag{3.2.21}$$

$$= \frac{E(\vec{p})+mc^2}{2mc^2}\frac{\left(E(\vec{p})+mc^2\right)^2+\vec{p}^2c^2}{\left(E(\vec{p})+mc^2\right)^2}$$

$$= \frac{2E^2(\vec{p})+2mc^2E(\vec{p})}{2mc^2\left(E(\vec{p})+mc^2\right)} = \frac{E(\vec{p})}{mc^2}$$

in agreement with $(3.1.20)$. For $r = 2$ and $r' = 3$ we have

$$\underline{w}_2^{\dagger}(\varepsilon_2\vec{p})\,\underline{w}_3(\varepsilon_3\vec{p}) = \frac{E(\vec{p})+mc^2}{2mc^2} \cdot \left(0, 1, \frac{c\left(p^{(1)}+\mathrm{i}p^{(2)}\right)}{E(\vec{p})+mc^2}, \frac{-cp^{(3)}}{E(\vec{p})+mc^2}\right) \begin{pmatrix} -cp^{(3)} \\ \dfrac{c\left(-p^{(1)}-\mathrm{i}p^{(2)}\right)}{E(\vec{p})+mc^2} \\ 1 \\ 0 \end{pmatrix} \tag{3.2.22}$$

$$\frac{E(\vec{p})+mc^2}{2mc^2}\left(-\frac{c\left(p^{(1)}+\mathrm{i}p^{(2)}\right)}{E(\vec{p})+mc^2}+\frac{c\left(p^{(1)}+\mathrm{i}p^{(2)}\right)}{E(\vec{p})+mc^2}\right)=0$$

according to $(3.2.20)$. The remaining combinations of r and r' can be dealt with similarly.

In quantum field theory instead of the box normalization above the "continuum normalization" or "normalization to delta functions" often is used. It is realized by the following modified functions

$$\underline{\varphi}_{\infty r}(\vec{x},t,\vec{p}) = \frac{1}{\sqrt{(2\pi\hbar)^3}}\sqrt{\frac{mc^2}{E(\vec{p})}}\,\underline{w}_r(\vec{p})\,e^{\mathrm{i}\varepsilon_r\left(\vec{p}\vec{x}-E(\vec{p})t\right)/\hbar}. \tag{3.2.23}$$

This function satisfies the Dirac equation $(3.1.6)$. It has the dimension $\left[\text{momentum}^{-3/2}\cdot\text{length}^{-3/2}\right]$. With the wave function $(3.2.23)$ we calculate

$$I=\int d^3x\,\underline{\varphi}^{\dagger}_{\infty r'}\left(\vec{x},t,\vec{p}'\right)\underline{\varphi}_{\infty r}\left(\vec{x},t,\vec{p}\right)=\int d^3x\frac{1}{\left(2\pi\hbar\right)^3}$$
$$\cdot\sqrt{\frac{m^2c^4}{E\left(\vec{p}'\right)E\left(\vec{p}\right)}}\,e^{\mathrm{i}\left(\varepsilon_{r'}E\left(\vec{p}'\right)-\varepsilon_r E\left(\vec{p}\right)\right)t/\hbar}\,e^{-\mathrm{i}\left(\varepsilon_{r'}\vec{p}'-\varepsilon_r\vec{p}\right)\vec{x}/\hbar}\,\underline{w}^{\dagger}_{r'}\left(\vec{p}'\right)\underline{w}_r\left(\vec{p}\right). \tag{3.2.24}$$

With the well-known mathematical relation

$$\int dx\frac{1}{2\pi}e^{-\mathrm{i}qx}=\delta\left(q\right)\ \text{ or }\ \int d^3x\frac{1}{\left(2\pi\right)^3}e^{-\mathrm{i}\vec{q}\vec{x}}=\delta^3\left(\vec{q}\right)$$
$$\text{i.e. }\int d^3x\,e^{-\mathrm{i}\vec{p}\vec{x}/\hbar}=\hbar^3\left(2\pi\right)^3\delta^3\left(\vec{p}\right) \tag{3.2.25}$$

we obtain $I=\dfrac{mc^2}{\hbar^3\sqrt{E\left(\vec{p}'\right)E\left(\vec{p}\right)}}e^{\mathrm{i}\left(\varepsilon_{r'}E\left(\vec{p}'\right)-\varepsilon_r E\left(\vec{p}\right)\right)t/\hbar}\hbar^3\delta^3\left(\varepsilon_{r'}\vec{p}'-\varepsilon_r\vec{p}\right)\underline{w}^{\dagger}_{r'}\left(\vec{p}'\right)\underline{w}_r\left(\vec{p}\right)$.

I.e. $\vec{p}'=\dfrac{\varepsilon_r}{\varepsilon_{r'}}\vec{p}=\varepsilon_r\varepsilon_{r'}\vec{p}$. Because of $\varepsilon_r^2=1$ and $E\left(\vec{p}\right)=E\left(-\vec{p}\right)$ we have $I=\dfrac{mc^2}{E\left(\vec{p}\right)}e^{\mathrm{i}\left(\varepsilon_{r'}-\varepsilon_r\right)E\left(\vec{p}\right)t/\hbar}\delta^3\left(\varepsilon_{r'}\vec{p}'-\varepsilon_r\vec{p}\right)\underline{w}^{\dagger}_{r'}\left(\varepsilon_{r'}\varepsilon_r\vec{p}\right)\underline{w}_r\left(\varepsilon_r\varepsilon_r\vec{p}\right)$. Denoting $\varepsilon_r\vec{p}$ by $\vec{p}''$ we write using $(3.2.20)$ $\underline{w}^{\dagger}_{r'}\left(\varepsilon_{r'}\vec{p}''\right)\underline{w}_r\left(\varepsilon_r\vec{p}''\right)=\delta_{rr'}\dfrac{E\left(\vec{p}''\right)}{mc^2}$ and finally

$$I=\int d^3x\,\underline{\varphi}^{\dagger}_{\infty r'}\left(\vec{x},t,\vec{p}'\right)\underline{\varphi}_{\infty r}\left(\vec{x},t,\vec{p}\right)=\delta_{rr'}\delta^3\left(\vec{p}'-\vec{p}\right). \tag{3.2.26}$$

This relation conveys that the function $(3.2.23)$ is "normalized to delta functions". Both sides of $(3.2.26)$ have the dimension $\left[\text{momentum}^{-3}\right]$.

The most general free particle solution of the Dirac equation $(3.1.5/6)$ is a linear combination of expressions $(3.2.23)$ which is integrated over $\vec{p}$ like this

$$\underline{\psi}_{\infty}\left(\vec{x},t\right)=\sum_{r=1}^{4}\int d^3p\,b_r\left(\vec{p}\right)\underline{\varphi}_{\infty r}\left(\vec{x},t,\vec{p}\right)$$
$$=\sum_{r=1}^{4}\int d^3p\frac{1}{\sqrt{\left(2\pi\hbar\right)^3}}\sqrt{\frac{mc^2}{E\left(\vec{p}\right)}}\,b_r\left(\vec{p}\right)\underline{w}_r\left(\vec{p}\right)e^{\mathrm{i}\varepsilon_r\left(\vec{p}\vec{x}-E\left(\vec{p}\right)t\right)/\hbar}. \tag{3.2.27}$$

This field must have the dimension $\left[\text{length}^{-3/2}\right]$. Therefore the dimension of $b_r\left(\vec{p}\right)$ is $\left[\text{momentum}^{-3/2}\right]$.

3.3 Quantum fields of Dirac particles

Similarly to $(2.2.2)$ the second quantization is performed by replacing $\underline{\psi}_\infty(\vec{x},t)$ and $b_r(\vec{p})$ in $(3.2.27)$ by operators as follows

$$\begin{aligned}\underline{\boldsymbol{\psi}}_\infty(\vec{x},t) &= \sum_{r=1}^{4}\int d^3p\, \boldsymbol{b}_r(\vec{p})\, \underline{\varphi}_{\infty r}(\vec{x},t,\vec{p}) \\ &= \sum_{r=1}^{4}\int d^3p\, \frac{1}{\sqrt{(2\pi\hbar)^3}}\sqrt{\frac{mc^2}{E(\vec{p})}}\boldsymbol{b}_r(\vec{p})\,\underline{w}_r(\vec{p})e^{i\varepsilon_r(\vec{p}\vec{x}-E(\vec{p})t)/\hbar}.\end{aligned} \quad (3.3.1)$$

In accordance with $(3.2.27)$ the operator function $\underline{\boldsymbol{\psi}}_\infty(\underline{x},t)$ has the dimension $\left[\text{length}^{-3/2}\right]$ and therefore the operator $\boldsymbol{b}_r(\vec{p})$ has the dimension $\left[\text{momentum}^{-3/2}\right]$. In analogy with $(2.2.3)$ the hermitean adjoint reads

$$\underline{\boldsymbol{\psi}}_\infty^\dagger(\vec{x}',t) = \sum_{r'=1}^{4}\int d^3p' \frac{1}{\sqrt{(2\pi\hbar)^3}}\sqrt{\frac{mc^2}{E(\vec{p}')}}\boldsymbol{b}_{r'}^\dagger(\vec{p}')\underline{w}_{r'}^\dagger(\vec{p}')e^{-i\varepsilon_{r'}(\vec{p}'\vec{x}'-E(\vec{p}')t)/\hbar}. \quad (3.3.2)$$

We postulate the following equal-time anticommutation rules analogously to $(2.3.1/2)$

$$\left[\boldsymbol{\psi}_{\infty\alpha}(\vec{x},t),\boldsymbol{\psi}^\dagger{}_{\infty\beta}(\vec{x}',t)\right]_+ = \delta_{\alpha\beta}\delta^3(\vec{x}-\vec{x}') \quad (3.3.3)$$

$$\left[\boldsymbol{\psi}_{\infty\alpha}(\vec{x},t),\boldsymbol{\psi}_{\infty\beta}(\vec{x}',t)\right]_+ = \left[\boldsymbol{\psi}^\dagger{}_{\infty\alpha}(\vec{x},t),\boldsymbol{\psi}^\dagger{}_{\infty\beta}(\vec{x}',t)\right]_+ = 0, \quad (3.3.4)$$

where $\boldsymbol{\psi}_{\infty\alpha}(\vec{x},t)$ and $\boldsymbol{\psi}_{\infty\beta}(\vec{x}',t)$ are components of the four-spinor $\underline{\boldsymbol{\psi}}_\infty(\vec{x},t)$.

Now we investigate the consequences of these rules to the operators $\boldsymbol{b}_r(\vec{p})$ and $\boldsymbol{b}^\dagger{}_{r'}(\vec{p}')$. We can isolate $\boldsymbol{b}_r(\vec{p})$ using $\underline{\boldsymbol{\psi}}_\infty(\vec{x},t)$, $(3.3.1)$, and the adjoint of $\underline{\varphi}_{\infty r}(\vec{x},t,\vec{p})$, $(3.2.18)$,

$$\begin{aligned}&\int d^3x \underline{\varphi}_{\infty r}^\dagger(\vec{x},t,\vec{p})\,\underline{\boldsymbol{\psi}}_\infty(\vec{x},t) \\ &= \sum_{r'=1}^{4}\int d^3p'\, \boldsymbol{b}_{r'}(\vec{p}')\int d^3x \underline{\varphi}_{\infty r}^\dagger(\vec{x},t,\vec{p})\,\underline{\varphi}_{\infty r'}(\vec{x},t,\vec{p}') \\ &= \sum_{r'=1}^{4}\int d^3p'\, \boldsymbol{b}_{r'}(\vec{p}')\,\delta_{rr'}{}^3\delta^3(\vec{p}-\vec{p}') = \boldsymbol{b}_r(\vec{p}),\end{aligned} \quad (3.3.5)$$

where $(3.2.26)$ has been applied. If we write down the components of the spinors this relation reads

$$\boldsymbol{b}_r(\vec{p}) = \int d^3x \sum_{\alpha=1}^{4}\varphi_{\infty r\alpha}^\dagger(\vec{x},t,\vec{p})\,\boldsymbol{\psi}_{\infty\alpha}(\vec{x},t). \quad (3.3.6)$$

Similarly the hermitean conjugate operator becomes

$$\boldsymbol{b}^{\dagger}_{r'}(\vec{p}') = \int d^3x' \underline{\boldsymbol{\psi}}^{\dagger}_{\infty}(\vec{x}',t)\,\underline{\varphi}_{\infty r'}(\underline{x}',t,\vec{p}') = \int d^3x' \sum_{\beta=1}^{4} \underline{\boldsymbol{\psi}}^{\dagger}_{\infty\beta}(\vec{x}',t)\,\underline{\varphi}_{\infty r'\beta}(\underline{x}',t,\vec{p}'). \tag{3.3.7}$$

Taking $(3.3.3)$ into account we compute the anticommutator

$$\begin{aligned}
&\left[\boldsymbol{b}_r(\vec{p}),\boldsymbol{b}^{\dagger}{}_{r'}(\vec{p}')\right]_+ \\
&= \int d^3x \int d^3x' \sum_{\alpha,\beta=1}^{4} \varphi^{\dagger}_{\infty r\alpha}(\vec{x},t,\vec{p})\,\varphi_{\infty r'\beta}(\vec{x}',t,\vec{p}')\left[\boldsymbol{\psi}_{\infty\alpha}(\vec{x},t),\boldsymbol{\psi}^{\dagger}_{\infty\beta}(\vec{x}',t)\right]_+ \\
&= \int d^3x \int d^3x' \sum_{\alpha,\beta=1}^{4} \varphi^{\dagger}_{\infty r\alpha}(\vec{x},t,\vec{p})\,\varphi_{\infty r'\beta}(\vec{x}',t,\vec{p}')\,\delta_{\alpha\beta}\delta^3(\vec{x}-\underline{x}') \\
&= \int d^3x \sum_{\alpha,=1}^{4} \varphi^{\dagger}_{\infty r\alpha}(\vec{x},t,\vec{p})\,\varphi_{\infty r'\alpha}(\vec{x},t,\vec{p}') \\
&= \int d^3x\, \underline{\varphi}^{\dagger}_{\infty r}(\vec{x},t,\vec{p})\,\underline{\varphi}_{\infty r'}(\vec{x},t,\vec{p}') = \delta_{rr'}\delta^3(\vec{p}-\vec{p}'),
\end{aligned} \tag{3.3.8}$$

where for the last step equation $(3.2.26)$ has been used. Similarly the remaining anticommutation relations can be deduced

$$\left[\boldsymbol{b}_r(\vec{p}),\boldsymbol{b}_{r'}(\vec{p}')\right]_+ = \left[\boldsymbol{b}^{\dagger}_r(\vec{p}),\boldsymbol{b}^{\dagger}_{r'}(\vec{p}')\right]_+ = 0 \tag{3.3.9}$$

in analogy with $(2.3.5)$.

Now the Hamilton operator of the quantized Dirac theory will be derived. We start from the quantized form of the Hamilton density $(3.1.15)$

$$\begin{aligned}
\boldsymbol{\mathcal{H}}(\vec{x},t) &= -\mathrm{i}\hbar c \sum_{k=1}^{4}\sum_{i=1}^{3} \boldsymbol{\psi}^{\dagger}_{\infty k}\left(\underline{\underline{\alpha}}_i \frac{d\boldsymbol{\psi}}{dx^i}\right)_k + mc^2 \sum_{k=1}^{4} \boldsymbol{\psi}^{\dagger}_{\infty k}\left(\underline{\underline{\beta}}\underline{\boldsymbol{\psi}}_{\infty}\right)_k \\
&= \underline{\boldsymbol{\psi}}^{\dagger}_{\infty}(\vec{x},t)\left(-\mathrm{i}\hbar c\,\underline{\underline{\vec{\alpha}}}\cdot\vec{\nabla} + mc^2\underline{\underline{\beta}}\right)\underline{\boldsymbol{\psi}}_{\infty}(\vec{x},t).
\end{aligned} \tag{3.3.10}$$

The quantized Hamiltonian itself reads using $(3.3.1)$

$$\begin{aligned}
\boldsymbol{H} &= \int d^3x\, \boldsymbol{\mathcal{H}}(\vec{x},t) = \int d^3x\, \underline{\boldsymbol{\psi}}^{\dagger}_{\infty}(\vec{x},t)\left(-\mathrm{i}\hbar c\,\underline{\underline{\vec{\alpha}}}\cdot\vec{\nabla} + mc^2\underline{\underline{\beta}}\right)\underline{\boldsymbol{\psi}}_{\infty}(\vec{x},t) \\
&= \sum_{r,r'=1}^{4} \int d^3p' d^3p\, \boldsymbol{b}^{\dagger}_{r'}(\vec{p}')\,\boldsymbol{b}_r(\vec{p}) \\
&\cdot \int d^3x\, \varphi^{\dagger}_{\infty r'}(\vec{x},t,\vec{p}')\left(-\mathrm{i}\hbar c\,\underline{\underline{\vec{\alpha}}}\cdot\vec{\nabla} + mc^2\underline{\underline{\beta}}\right)\varphi_{\infty r}(\vec{x},t,\vec{p}).
\end{aligned} \tag{3.3.11}$$

Since the plane wave $\varphi_{\infty r}(\vec{x},t,\vec{p}) = \frac{1}{\sqrt{(2\pi\hbar)^3}}\sqrt{\frac{mc^2}{E(\vec{p})}}\underline{w}_r(\vec{p})e^{\mathrm{i}\varepsilon_r(\vec{p}\vec{x}-E(\vec{p})t)/\hbar}$ obeys the Dirac equation $(3.1.6)$ we have

$$\left(-\mathrm{i}\hbar c\underline{\underline{\alpha}}\cdot\vec{\nabla}+mc^2\underline{\underline{\beta}}\right)\underline{\varphi}_{\infty r}\left(\vec{x},t,\vec{p}\right)=\mathrm{i}\hbar\underline{\dot{\varphi}}_{\infty r}\left(\vec{x},t,\vec{p}\right)=\varepsilon_r E\left(\vec{p}\right)\underline{\varphi}_{\infty r}\left(\vec{x},t,\vec{p}\right) \quad (3.3.12)$$

and due to the orthogonality relation $(3.2.26)$ the Hamilton operator reads

$$\begin{aligned}\boldsymbol{H} &= \sum_{r,r'=1}^{4}\int d^3p'\,d^3p\,\boldsymbol{b}_{r'}^{\dagger}\left(\vec{p}\,'\right)\boldsymbol{b}_r\left(\vec{p}\right)\varepsilon_r E\left(\vec{p}\right)\\ &\quad\cdot\int d^3x\underline{\varphi}_{\infty r'}^{\dagger}\left(\vec{x},t,\vec{p}\,'\right)\underline{\varphi}_{\infty r}\left(\vec{x},t,\vec{p}\right)\\ &= \sum_{r=1}^{4}\int d^3p\,\varepsilon_r E\left(\vec{p}\right)\boldsymbol{b}_r^{\dagger}\left(\vec{p}\right)\boldsymbol{b}_r\left(\vec{p}\right).\end{aligned} \quad (3.3.13)$$

Starting from this expression, similar to the proceeding in section 2.2, we can find particle number operators and ascertain the properties of the operators $\boldsymbol{b}_r\left(\vec{p}\right)$ and $\boldsymbol{b}_{r'}^{\dagger}\left(\vec{p}\,'\right)$. The eigenvalue of $\boldsymbol{H}$ of a multi-particle state is its total energy. It is the sum of the energies $E\left(\vec{p}\right)$ of the particles. Therefore, for $r=1$ and 2 the operator $d^3p\,\boldsymbol{b}_r^{\dagger}\left(\vec{p}\right)\boldsymbol{b}_r\left(\vec{p}\right)$ in $(3.3.13)$ represents the number of particles in the area d^3p for these r-values. In other words, there exists a differential number operator

$$\boldsymbol{n}_r\left(\vec{p}\right)=\boldsymbol{b}_r^{\dagger}\left(\vec{p}\right)\boldsymbol{b}_r\left(\vec{p}\right)\ \text{ for } r=1 \text{ or } 2. \quad (3.3.14)$$

This interpretation cannot be valid for $r=3$ and 4. Because of $\varepsilon_r=-1$ for these r-values the total energy could become negative, which does not make sense for free particles. In the partial sum of $(3.3.13)$ with $r=3$ and 4 we insert $(3.3.8)$ like this

$$\begin{aligned}\boldsymbol{H}_{3,4} &= \sum_{r=3}^{4}\int d^3p\,\varepsilon_r E\left(\vec{p}\right)\boldsymbol{b}_r^{\dagger}\left(\vec{p}\right)\boldsymbol{b}_r\left(\vec{p}\right)\\ &= -\sum_{r=3}^{4}\int d^3p\,E\left(\vec{p}\right)\left(\delta(0)-\boldsymbol{b}_r\left(\vec{p}\right)\boldsymbol{b}_r^{\dagger}\left(\vec{p}\right)\right).\end{aligned} \quad (3.3.15)$$

Because

$$-\sum_{r=3}^{4}\int d^3pE\left(\vec{p}\right)\delta(0) \quad (3.3.16)$$

is extremely divergent and since we are only interested in energy differences we perform a renormalization and leave out the term $(3.3.16)$. Then the renormalized energy operator $\boldsymbol{H}'$ reads

$$\boldsymbol{H}'=\sum_{r=1}^{2}\int d^3pE\left(\vec{p}\right)\boldsymbol{b}_r^{\dagger}\left(\vec{p}\right)\boldsymbol{b}_r\left(\vec{p}\right)+\sum_{r=3}^{4}\int d^3pE\left(\vec{p}\right)\boldsymbol{b}_r\left(\vec{p}\right)\boldsymbol{b}_r^{\dagger}\left(\vec{p}\right) \quad (3.3.17)$$

with positive $E\left(\vec{p}\right)$ according to $(3.2.2)$. Therefore, the differential number operator for $r=3$ or 4 reads

$$\boldsymbol{n}_r\left(\vec{p}\right)=\boldsymbol{b}_r\left(\vec{p}\right)\boldsymbol{b}_r^{\dagger}\left(\vec{p}\right)\ \text{ for } r=3 \text{ or } 4. \quad (3.3.18)$$

We go back to the cases $r=1$ and 2. The differential number operator $(3.3.14)$ must generate the eigenvalue 0 if it acts on the vacuum state $|0\rangle$

$$\boldsymbol{n}_r(\vec{p})|0\rangle=\boldsymbol{b}_r^\dagger(\vec{p})\boldsymbol{b}_r(\vec{p})|0\rangle=0 \quad \text{for } r=1 \text{ and } 2. \tag{3.3.19}$$

This is fulfilled if

$$\boldsymbol{b}_r(\vec{p})|0\rangle=0 \quad \text{for } r=1 \text{ and } 2 \tag{3.3.20}$$

holds analogously to $(2.2.24)$. As in section 2.2, $\boldsymbol{b}_r(\vec{p})$ is named annihilation operator. Moreover, we show that $\boldsymbol{b}_r(\vec{p})$ annihilates a particle in a many-particles state $|\Phi\rangle$. First we apply the total number operator $\int d^3p'\boldsymbol{b}_r^\dagger(\vec{p}')\boldsymbol{b}_r(\vec{p}')$ on $|\Phi\rangle$. The eigenvalue of this operation is the particle number N_r $(r=1 \text{ or } 2)$. If the number operator acts on the state $\boldsymbol{b}_r(\vec{p})|\Phi\rangle$ we obtain using $(3.3.8)$ and $(3.3.9)$

$$\begin{aligned}
&\int d^3p'\,\boldsymbol{b}_r^\dagger(\vec{p}')\boldsymbol{b}_r(\vec{p}')\boldsymbol{b}_r(\vec{p})|\Phi\rangle\\
&=-\int d^3p'\boldsymbol{b}_r^\dagger(\vec{p}')\boldsymbol{b}_r(\vec{p})\boldsymbol{b}_r(\vec{p}')|\Phi\rangle\\
&=\int d^3p'\left(\boldsymbol{b}_r(\vec{p})\boldsymbol{b}_r^\dagger(\vec{p}')\boldsymbol{b}_r(\vec{p}')-\delta^3(\vec{p}-\vec{p}')\boldsymbol{b}_r(\vec{p}')\right)|\Phi\rangle\\
&=\boldsymbol{b}_r(\vec{p})\int d^3p'\,\boldsymbol{b}_r^\dagger(\vec{p}')\boldsymbol{b}_r(\vec{p}')|\Phi\rangle-\boldsymbol{b}_r(\vec{p})|\Phi\rangle\\
&=(N_r-1)\boldsymbol{b}_r(\vec{p})|\Phi\rangle \quad (r=1 \text{ or } 2).
\end{aligned} \tag{3.3.21}$$

Thus, the state $\boldsymbol{b}_r(\vec{p})|\Phi\rangle$ has one particle less than $|\Phi\rangle$ as we expect from the annihilation operator $\boldsymbol{b}_r(\vec{p})$, $(3.3.20)$. Analogously it can be shown that $\boldsymbol{b}_r^\dagger(\vec{p})$ (for r=1 or 2) is a creation operator.

We come back to the types $r=3$ and 4 and look into the action of the operator $\boldsymbol{b}_r^\dagger(\vec{p})$ on a many particle state $|\Phi\rangle$. Here, corresponding to $(3.3.18)$, the total number operator reads $\int d^3p'\boldsymbol{b}_r(\vec{p}')\boldsymbol{b}_r^\dagger(\vec{p}')$ with the eigenvalue N_r of $|\Phi\rangle$. We apply this operator on the state $\boldsymbol{b}_r^\dagger(\vec{p})|\Phi\rangle$, $r=3$ or 4.

$$\begin{aligned}
&\int d^3p'\boldsymbol{b}_r(\vec{p}')\boldsymbol{b}_r^\dagger(\vec{p}')\boldsymbol{b}_r^\dagger(\vec{p})|\Phi\rangle\\
&=-\int d^3p'\boldsymbol{b}_r(\vec{p}')\boldsymbol{b}_r^\dagger(\vec{p})\boldsymbol{b}_r^\dagger(\vec{p}')|\Phi\rangle\\
&=\int d^3p'\left(\boldsymbol{b}_r^\dagger(\vec{p})\boldsymbol{b}_r(\vec{p}')-\delta^3(\vec{p}-\vec{p}')\right)\boldsymbol{b}_r^\dagger(\vec{p}')|\Phi\rangle\\
&=\boldsymbol{b}_r^\dagger(\vec{p})N_r|\Phi\rangle-\boldsymbol{b}_r^\dagger(\vec{p})N_r|\Phi\rangle=(N_r-1)\boldsymbol{b}_r^\dagger(\vec{p})|\Phi\rangle \qquad (r=3 \text{ or } 4).
\end{aligned} \tag{3.3.22}$$

Thus, for $r=3$ and 4 the operator $\boldsymbol{b}_r^\dagger(\vec{p})$ is annihilating. Analogously one shows that $\boldsymbol{b}_r(\vec{p})$ is a creation operator for $r=3$ and 4. The roles of the operators $\boldsymbol{b}_r^\dagger(\vec{p})$ and $\boldsymbol{b}_r(\vec{p})$ are interchanged relative to the situation with $r=1$ and 2.

Correspondingly the particles which are generated by $\boldsymbol{b}_r(\vec{p})$ with $r = 3$ or 4 are named antiparticles. The antiparticles of the electrons are positrons.

In order to take into account the behaviour for r = 3 and 4 we rename the operators $\boldsymbol{b}$ by $\boldsymbol{d}^\dagger$ and the indices 3 and 4 are replaced by 1 and 2 like this

$$\begin{aligned} \boldsymbol{b}_{r=3}(\vec{p}) &= \boldsymbol{d}^\dagger_{r=1}(\vec{p}) \\ \boldsymbol{b}_{r=4}(\vec{p}) &= \boldsymbol{d}^\dagger_{r=2}(\vec{p}) \quad (\text{creation operators}) \end{aligned} \tag{3.3.23}$$

The hermitean conjugate form $\boldsymbol{d}^\dagger$ is chosen for sake of convenience. Consequently we have

$$\begin{aligned} \boldsymbol{b}^\dagger_{r=3}(\vec{p}) &= \boldsymbol{d}_{r=1}(\vec{p}) \\ \boldsymbol{b}^\dagger_{r=4}(\vec{p}) &= \boldsymbol{d}_{r=2}(\vec{p}) \quad (\text{annihilation operators}). \end{aligned} \tag{3.3.24}$$

We insert $(3.3.23)$ and $(3.3.24)$ in the renormalized energy operator $\boldsymbol{H}'$, $(3.3.17)$, which yields

$$\boldsymbol{H}' = \sum_{r=1}^{2} \int d^3p\, E(\vec{p}) \left(\boldsymbol{b}^\dagger_r(\vec{p}) \boldsymbol{b}_r(\vec{p}) + \boldsymbol{d}^\dagger_r(\vec{p}) \boldsymbol{d}_r(\vec{p}) \right), \tag{3.3.25}$$

where the $\boldsymbol{b}$'s and the $\boldsymbol{d}$'s form the same structure. We point out that $\boldsymbol{d}^\dagger_r(\vec{p})$ is a creator of a $r = 3(4)$-particle and $\boldsymbol{d}_r(\vec{p})$ an annihilator (as $\boldsymbol{b}_r(\vec{p})$ in $(3.3.21)$). Corresponding to $(3.3.14)$ the differential number operator for antiparticles reads

$$\boldsymbol{n}_r(\vec{p}) = \boldsymbol{d}^\dagger_r(\vec{p}) \boldsymbol{d}_r(\vec{p}) \quad \text{r = 1 or 2}. \tag{3.3.26}$$

From $(3.3.8)$ and $(3.3.23)$ we take

$$\left[\boldsymbol{d}^\dagger_r(\vec{p}), \boldsymbol{d}_{r'}(\vec{p}\,') \right]_+ = \delta_{rr'} \delta^3(\vec{p} - \vec{p}\,'), \qquad r', r = 1 \text{ or } 2. \tag{3.3.27}$$

Other anticommutators containing $\boldsymbol{d}, \boldsymbol{d}^\dagger$ or a $\boldsymbol{d}$-$\boldsymbol{b}$-pair vanish. For r = 3 and 4 not only the operators are renamed but also the coefficients (spinors) $\underline{w}_r(\vec{p})$, $(3.2.19)$, get new markings like this

$$\begin{aligned} \underline{w}_3(\vec{p}) &= \underline{v}_1(\vec{p}) \\ \underline{w}_4(\vec{p}) &= \underline{v}_2(\vec{p}). \end{aligned} \tag{3.3.28}$$

Therefore, the field operator $\underline{\boldsymbol{\psi}}(\vec{x}, t)$, $(3.3.1)$, now is written as follows

$$\underline{\boldsymbol{\psi}}(\vec{x},t) = \sum_{r=1}^{2}\int \frac{d^3p}{(2\pi\hbar)^{3/2}}\sqrt{\frac{mc^2}{E(\vec{p})}}$$
$$\cdot\left(\boldsymbol{b}_r(\vec{p})\underline{w}_r(\vec{p})e^{\mathrm{i}(\vec{p}\vec{x}-E(\vec{p})t)/\hbar} + \boldsymbol{d}_r^\dagger(\vec{p})\underline{v}_r(\vec{p})e^{-\mathrm{i}(\vec{p}\vec{x}-E(\vec{p})t)/\hbar}\right)$$

and (3.3.29)

$$\underline{\boldsymbol{\psi}}^\dagger(\vec{x},t) = \sum_{r=1}^{2}\int \frac{d^3p}{(2\pi\hbar)^{3/2}}\sqrt{\frac{mc^2}{E(\vec{p})}}$$
$$\cdot\left(\boldsymbol{b}^\dagger{}_r(\vec{p})\underline{w}_r(\vec{p})e^{-\mathrm{i}(\vec{p}\vec{x}-E(\vec{p})t)/\hbar} + \boldsymbol{d}_r(\vec{p})\underline{v}_r(\vec{p})e^{\mathrm{i}(\vec{p}\vec{x}-E(\vec{p})t)/\hbar}\right).$$

These field functions are normalized using the continuum normalization as in $(3.3.1)$. Due to $(3.3.1)$ the operators $\boldsymbol{b}_r(\vec{p})$ and consequently $\boldsymbol{d}_r(\vec{p})$ have the dimension $\left[\text{momentum}^{-3/2}\right]$. Therefore, the operator $\underline{\boldsymbol{\psi}}(\vec{x},t)$ has the dimension $\left[\text{length}^{-3/2}\right]$ as we expect.

One can go back to box normalization by replacing the factor $(2\pi\hbar)^{3/2}$ in $(3.3.29)$ by $\sqrt{V}$ according to $(3.2.18)$. In order to obtain again the dimension $\left[\text{length}^{-3/2}\right]$ for $\underline{\boldsymbol{\psi}}(\vec{x},t)$ one has to multiply $\sqrt{V}$ by a normalization factor $\mathcal{E}$, which shows dimension $\left[\text{momentum}^{3/2}\right]$:

$$\frac{1}{(2\pi\hbar)^{3/2}} \;\rightarrow\; \frac{1}{\mathcal{E}\sqrt{V}} \quad \text{for box normalization.} \tag{3.3.30}$$

In $(3.3.15)$ up to $(3.3.17)$ we have renormalized the energy operator $\boldsymbol{H}$ in order to obtain positive and finite eigenvalues for free particles. The same result as $(3.3.25)$ arises if the procedure of **normal ordering** is applied to products of the operators $\boldsymbol{b}$ or $\boldsymbol{d}$ with $\boldsymbol{b}^\dagger$ or $\boldsymbol{d}^\dagger$. Normal ordering of such products demands by definition that in all such products containing a hermitean conjugate operator on the right of a non conjugate one the factors (operators) have to be interchanged and the sign has to be changed (this $(+-)$-change is a speciality of Fermi particles). This means that the products $\boldsymbol{bd}, \boldsymbol{b}^\dagger\boldsymbol{b}, \boldsymbol{b}^\dagger\boldsymbol{d}, \boldsymbol{b}^\dagger\boldsymbol{d}^\dagger$ etc remain unchanged but $\boldsymbol{bb}^\dagger$, $\boldsymbol{bd}^\dagger$, $\boldsymbol{db}^\dagger$, and $\boldsymbol{dd}^\dagger$ have to be changed.

We apply this procedure to $\boldsymbol{H}$, $(3.3.13)$, which reads

$$\boldsymbol{H} = \int d^3p E(\vec{p})\left(\sum_{r=1}^{2}\boldsymbol{b}_r^\dagger(\vec{p})\boldsymbol{b}_r(\vec{p}) - \sum_{r=3}^{4}\boldsymbol{b}_r^\dagger(\vec{p})\boldsymbol{b}_r(\vec{p})\right).$$

Inserting $(3.3.23\,/\,24)$ yields

$$\boldsymbol{H} = \int d^3p\, E(\vec{p})\sum_{r=1}^{2}\left(\boldsymbol{b}_r^\dagger(\vec{p})\boldsymbol{b}_r(\vec{p}) - \boldsymbol{d}_r(\vec{p})\boldsymbol{d}_r^\dagger(\vec{p})\right). \tag{3.3.31}$$

We apply normal ordering to $\boldsymbol{H}$, which is marked by two colons

$$
\begin{aligned}
:\boldsymbol{H}: &= \int d^3p\, E(\vec{p}) \sum_{r=1}^{2} \left(:\boldsymbol{b}_r^\dagger(\vec{p})\boldsymbol{b}_r(\vec{p}): - :\boldsymbol{d}_r(\vec{p})\boldsymbol{d}_r^\dagger(\vec{p}): \right) \\
&= \int d^3p\, E(\vec{p}) \sum_{r=1}^{2} \left(\boldsymbol{b}_r^\dagger(\vec{p})\boldsymbol{b}_r(\vec{p}) + \boldsymbol{d}_r^\dagger(\vec{p})\boldsymbol{d}_r(\vec{p}) \right)
\end{aligned} \tag{3.3.32}
$$

which is identical with $\boldsymbol{H}'$ in $(3.3.25)$. Normal ordering is a formal tool which renormalizes operators like those for energy, charge, angular momentum etc.. It appears also in Wick's theorem (section 5.4).

3.4 The Feynman propagator for Dirac fields

In Wick's theorem so-called contractions of field operators play an important rôle. The contraction of the field operators $\boldsymbol{\psi}_\alpha$ and $\left(\underline{\boldsymbol{\psi}}^\dagger \cdot \underline{\underline{\beta}}\right)_\beta$ by definition is identical with the Feynman propagator for spin-1/2 particles, which we calculate now.

We start with the component $\boldsymbol{\psi}_\alpha(\vec{x},t)$ of the field operator $\underline{\boldsymbol{\psi}}(\vec{x},t)$, $(3.3.1)$ or $(3.3.29)$, and the component $\boldsymbol{\psi}_\beta^\dagger(\vec{x},t)$ of $\underline{\boldsymbol{\psi}}^\dagger(\vec{x},t)$, $(3.3.2)$, which we modify as follows: we define $\bar{\boldsymbol{\psi}}_\beta(\vec{x},t) = -\boldsymbol{\psi}_\beta^\dagger(\vec{x},t)$ for $\beta = 3$ and 4 and $\bar{\boldsymbol{\psi}}_\beta(\vec{x},t) = \boldsymbol{\psi}_\beta^\dagger(\vec{x},t)$ for $\beta = 1$ and 2. The operator (spinor) can just be formed as $\underline{\bar{\boldsymbol{\psi}}} = \underline{\boldsymbol{\psi}}^\dagger\, \underline{\underline{\beta}}$ with the matrix $\underline{\underline{\beta}}$, $(3.1.2)$, i.e.

$$
\bar{\boldsymbol{\psi}}_\beta(\vec{x}',t') = \left(\underline{\boldsymbol{\psi}}^\dagger(\vec{x}',t') \cdot \underline{\underline{\beta}} \right)_\beta . \tag{3.4.1}
$$

We introduce the time-ordered product of field operators taken at different points in space and time which is defined as follows

$$
T\left(\boldsymbol{\psi}_\alpha(\vec{x},t)\bar{\boldsymbol{\psi}}_\beta(\vec{x}',t')\right) = \left\{ \begin{array}{ll} \boldsymbol{\psi}_\alpha(\vec{x},t)\bar{\boldsymbol{\psi}}_\beta(\vec{x}',t') & \text{for } t > t' \\ -\bar{\boldsymbol{\psi}}_\beta(\vec{x}',t')\boldsymbol{\psi}_\alpha(\vec{x},t) & \text{for } t' > t \end{array} \right\}. \tag{3.4.2}
$$

The minus sign accounts for the fermionic (anticommutating) character of the field operators. In this section we will use the relativistic four-dimensional coordinates

$$
\underline{x} = \left(x^0 = ct, x^{(1)} = x, x^{(2)} = y, x^{(3)} = z \right) \tag{3.4.3}
$$

and define the Feynman propagator $S_{F,\alpha\beta}(\underline{x} - \underline{x}')$ like this

$$
\mathrm{i}S_{F,\alpha\beta}(\underline{x} - \underline{y}) = \langle 0 | T\left(\boldsymbol{\psi}_\alpha(\underline{x})\bar{\boldsymbol{\psi}}_\beta(\underline{y})\right) | 0 \rangle . \tag{3.4.4}
$$

In fact, $(3.4.4)$ is the vacuum expectation value of the time-ordered product of field operators taken at different points in space-time. With $(3.4.2)$ and $(3.4.4)$ we can write

$$\mathrm{i}S_{F,\alpha\beta} = \begin{Bmatrix} \langle 0|\boldsymbol{\psi}_\alpha(\underline{x})\bar{\boldsymbol{\psi}}_\beta(\underline{y})|0\rangle & \text{for } \mathrm{x}^0 > y^0 \\ -\langle 0|\bar{\boldsymbol{\psi}}_\beta(\underline{y})\boldsymbol{\psi}_\alpha(\underline{x})|0\rangle & \text{for } y^0 > \mathrm{x}^0 \end{Bmatrix}. \tag{3.4.5}$$

The definitions $(3.4.4)$ and $(3.4.5)$ result in

$$\langle 0|T\left(\bar{\boldsymbol{\psi}}_\beta(\underline{y})\boldsymbol{\psi}_\alpha(\underline{x})\right)|0\rangle = -\langle 0|\boldsymbol{\psi}_\alpha(\underline{x})\bar{\boldsymbol{\psi}}_\beta(\underline{y})|0\rangle = -\mathrm{i}S_{F,\alpha\beta}\,.$$

We remind that, since $\boldsymbol{b}_r(\vec{p})$ and $\boldsymbol{d}_r(\vec{p})$ are annihilation operators, we have

$$\boldsymbol{b}_r(\vec{p})|0\rangle = \boldsymbol{d}_r(\vec{p})|0\rangle = 0 \tag{3.4.6}$$

and analogously for the hermitean adjoint operators

$$\langle 0|\boldsymbol{b}_r^\dagger(\vec{p}) = \langle 0|\boldsymbol{d}_r^\dagger(\vec{p}) = 0\,. \tag{3.4.7}$$

From now on we will use the relativistic formulation (notice the minus sign!)

$$\begin{aligned} &\underline{p}\cdot\underline{x} = p^0 x^0 - \vec{p}\vec{x} \\ &\text{with } \underline{p} = \left(p^0,\vec{p}\right) = \left(p^0,p^{(1)},p^{(2)},p^{(3)}\right),\ p^0 = E(\vec{p})/\mathrm{c} \\ &\text{and } \mathrm{x}^0 = ct. \end{aligned} \tag{3.4.8}$$

With the aid of $(3.3.29)$ and $(3.4.6)$ up to $(3.4.8)$ we obtain

$$\begin{aligned} \langle 0|\boldsymbol{\psi}_\alpha(\underline{x})\bar{\boldsymbol{\psi}}_\beta(\underline{y})|0\rangle = \langle 0|&\sum_{r=1}^{2}\int\frac{d^3p}{(2\pi\hbar)^{3/2}}\sqrt{\frac{mc^2}{E(\vec{p})}}\boldsymbol{b}_r(\vec{p})w_{r\alpha}(\vec{p})e^{-\mathrm{i}\underline{p}\underline{x}/\hbar} \\ &\cdot\sum_{r'=1}^{2}\int\frac{d^3p'}{(2\pi\hbar)^{3/2}}\sqrt{\frac{mc^2}{E(\vec{p}\,')}}\boldsymbol{b}_{r'}^\dagger(\vec{p}\,')\bar{w}_{r'\beta}(\vec{p}\,')e^{\mathrm{i}\underline{p}'\underline{y}/\hbar}|0\rangle \end{aligned} \tag{3.4.9}$$

with $\bar{w}_{r'\beta}(\vec{p}\,') \equiv \left(\underline{w}_{r'}^\dagger(\vec{p}\,')\cdot\underline{\underline{\beta}}\right)_\beta$ i.e. $\underline{\bar{w}}_{r'}(\vec{p}\,') \equiv \underline{w}_{r'}^\dagger(\vec{p}\,')\underline{\underline{\beta}}$.

Due to $(3.3.8)$ there holds

$$\begin{aligned} &\langle 0|\boldsymbol{b}_r(\vec{p})\boldsymbol{b}_{r'}^\dagger(\vec{p}\,')|0\rangle \\ &= \langle 0|\left(-\boldsymbol{b}_{r'}^\dagger(\vec{p}\,')\boldsymbol{b}_r(\vec{p}) + \delta_{rr'}\delta^3(\vec{p}-\vec{p}\,')\right)|0\rangle = \delta_{rr'}\delta^3(\vec{p}-\vec{p}\,'), \end{aligned} \tag{3.4.10}$$

which we insert in $(3.4.9)$:

$$\langle 0|\,\boldsymbol{\psi}_\alpha(\underline{x})\,\bar{\boldsymbol{\psi}}_\beta(\underline{y})|0\rangle = \int\frac{d^3p}{(2\pi\hbar)^3}\frac{mc^2}{E(\vec{p})}e^{-\mathrm{i}\underline{p}(\underline{x}-\underline{y})/\hbar}\sum_{r=1}^{2}w_{r\alpha}(\vec{p})\,\bar{w}_{r\beta}(\vec{p}). \tag{3.4.11}$$

We now show that the sum in $(3.4.11)$ can be written in a compact form. Using the “Feynman dagger”

$$\underline{\underline{\not p}} \equiv \underline{\underline{\beta}}\left(p^0 - \sum_{i=1}^{3} \underline{\underline{\alpha}}_i p^{(i)} \right) \qquad (3.4.12)$$

we claim
$$\sum_{r=1}^{2} w_{r\alpha}(\vec{p})\,\bar{w}_{r\beta}(\vec{p}) = \left(\frac{c\underline{\underline{\not p}} + mc^2\,\underline{\underline{1}}}{2mc^2} \right)_{\alpha\beta} . \qquad (3.4.13)$$

The 4×4-matrix which underlies the right hand side of $(3.4.13)$ can be given in detail by means of $(3.1.2)$

$$\frac{c\underline{\underline{\not p}} + mc^2\,\underline{\underline{1}}}{2mc^2} = \frac{1}{2mc^2}\begin{pmatrix} E(\vec{p})+mc^2 & 0 & -cp^{(3)} & -c\left(p^{(1)}-\mathrm{i}p^{(2)}\right) \\ 0 & E(\vec{p})+mc^2 & -c\left(p^{(1)}+\mathrm{i}p^{(2)}\right) & cp^{(3)} \\ cp^{(3)} & c\left(p^{(1)}-\mathrm{i}p^{(2)}\right) & -E(\vec{p})+mc^2 & 0 \\ c\left(p^{(1)}+\mathrm{i}p^{(2)}\right) & -cp^{(3)} & 0 & -E(\vec{p})+mc^2 \end{pmatrix} \qquad (3.4.14)$$

In order to check the relation $(3.4.13)$ we calculate some matrix elements of the left hand side using $(3.2.19)$

$$\sum_{r=1}^{2} w_{r1}(\vec{p})\,\bar{w}_{r1}(\vec{p}) = \frac{E(\vec{p})+mc^2}{2mc^2},$$

which is in agreement with the $(1,1)$-element of the matrix $(3.4.14)$. Furthermore

$$\sum_{r=1}^{2} w_{r1}(\vec{p})\,w_{r2}(\vec{p}) = 0$$
$$\sum_{r=1}^{2} w_{r1}(\vec{p})\,w_{r3}(\vec{p}) = -\frac{cp^{(3)}}{2mc^2} + 0$$
$$\sum_{r=1}^{2} w_{r2}(\vec{p})\,w_{r3}(\vec{p}) = 0 - \frac{c\left(p^{(1)}+\mathrm{i}p^{(2)}\right)}{2mc^2}$$
$$\sum_{r=1}^{2} w_{r3}(\vec{p})\,w_{r3}(\vec{p}) = -\frac{c^2\left(p^{(3)}\right)^2 + c^2\left(p^{(1)}-\mathrm{i}p^{(2)}\right)\left(p^{(1)}+\mathrm{i}p^{(2)}\right)}{2mc^2\left(E(\vec{p})+mc^2\right)}$$
$$= -\frac{c^2\vec{p}^2}{2mc^2\left(E(\vec{p})+mc^2\right)} = -\frac{E(\vec{p})^2 - m^2c^4}{2mc^2\left(E(\vec{p})+mc^2\right)} = \frac{-E(\vec{p})+mc^2}{2mc^2}$$

etc. Analogously the following relation corresponding to $(3.4.13)$ can be shown (see $(3.3.28)$)

$$\sum_{r=1}^{2} v_{r,\alpha}(\vec{p})\,\bar{v}_{r,\beta}(\vec{p}) = \left(\frac{c\underline{\underline{\not p}} - mc^2\,\underline{\underline{1}}}{2mc^2}\right)_{\alpha\beta} . \tag{3.4.15}$$

In the same way as $(3.4.11)$ one can show

$$\begin{aligned} &\langle 0|\,\bar{\boldsymbol{\psi}}_{\beta}(\underline{y})\,\boldsymbol{\psi}_{\alpha}(\underline{x})|0\rangle \\ &= \int \frac{d^3p}{(2\pi\hbar)^3}\frac{mc^2}{E(\vec{p})} e^{\mathrm{i}\underline{p}(\underline{x}-\underline{y})/\hbar} \sum_{r=1}^{2} v_{r,\alpha}(\vec{p})\,\bar{v}_{r,\beta}(\vec{p}). \end{aligned} \tag{3.4.16}$$

We insert $(3.4.11), (3.4.13), (3.4.15)$ and $(3.4.16)$ in $(3.4.5)$ which yields

$$\mathrm{i}S_{F,\alpha\beta}(\underline{x}-\underline{y}) = \begin{Bmatrix} \displaystyle\int \frac{d^3p}{(2\pi\hbar)^3}\frac{1}{2E(\vec{p})} e^{-\mathrm{i}\underline{p}(\underline{x}-\underline{y})/\hbar}\left(c\underline{\underline{\not p}} + mc^2\,\underline{\underline{1}}\right)_{\alpha\beta} & \text{for } \mathrm{x}^0 > y^0 \\ \displaystyle -\int \frac{d^3p}{(2\pi\hbar)^3}\frac{1}{2E(\vec{p})} e^{\mathrm{i}\underline{p}(\underline{x}-\underline{y})/\hbar}\left(c\underline{\underline{\not p}} - mc^2\,\underline{\underline{1}}\right)_{\alpha\beta} & \text{for } y^0 > x^0 \end{Bmatrix}. \tag{3.4.17}$$

We introduce the following operator (nabla-dagger)

$$\underline{\underline{\not\nabla}}_{\underline{x}} = \underline{\underline{\beta}}\left(\frac{d}{dx^0} + \sum_{i=1}^{3}\underline{\underline{\alpha}}_i\,\frac{d}{dx^{(i)}}\right) \quad \text{(notice the plus sign)}, \tag{3.4.18}$$

which causes

$$\begin{aligned} &c\hbar\left(\mathrm{i}\underline{\underline{\not\nabla}}_{\underline{x}} + \frac{mc}{\hbar}\underline{\underline{1}}\right)_{\alpha\beta} e^{-\mathrm{i}\underline{p}(\underline{x}-\underline{y})/\hbar} \\ &= c\hbar\left(\mathrm{i}\underline{\underline{\beta}}\frac{d}{dx^0} + \mathrm{i}\underline{\underline{\beta}}\sum_{i=1}^{3}\underline{\underline{\alpha}}_i\,\frac{d}{dx^{(i)}} + \frac{mc\underline{\underline{1}}}{\hbar}\right)_{\alpha\beta} e^{-\mathrm{i}\left(p^0x^0 - \sum_{i=1}^{3}p^{(i)}x^{(i)}\right)/\hbar} e^{\mathrm{i}\underline{p}\underline{y}/\hbar} \\ &= c\left(\underline{\underline{\beta}}p^0 - \underline{\underline{\beta}}\sum_{i=1}^{3}\underline{\underline{\alpha}}_i p^{(i)} + mc\underline{\underline{1}}\right)_{\alpha\beta} e^{-\mathrm{i}\left(p^0x^0 - \sum_{i=1}^{3}p^{(i)}x^{(i)}\right)/\hbar} e^{\mathrm{i}\underline{p}\underline{y}/\hbar} \\ &= \left(c\underline{\underline{\not p}} + mc^2\,\underline{\underline{1}}\right)_{\alpha\beta} e^{-\mathrm{i}\underline{p}(\underline{x}-\underline{y})/\hbar}. \end{aligned} \tag{3.4.19}$$

Furthermore, $c\hbar\left(\mathrm{i}\underline{\underline{\not\nabla}}_{\underline{x}} + \frac{mc}{\hbar}\underline{\underline{1}}\right)_{\alpha\beta} e^{\mathrm{i}\underline{p}(\underline{x}-\underline{y})/\hbar} = \left(-c\underline{\underline{\not p}} + mc^2\,\underline{\underline{1}}\right)_{\alpha\beta} e^{\mathrm{i}\underline{p}(\underline{x}-\underline{y})/\hbar}.$ (3.4.20)

The expressions $(3.4.19)$ and $(3.4.20)$ are inserted in $(3.4.17)$ like this

$$\mathrm{i}S_{F,\alpha\beta}(\underline{x}-\underline{y}) = \left(\mathrm{i}\underline{\underline{\not\nabla}}_{\underline{x}} + \frac{mc}{\hbar}\underline{\underline{1}}\right)_{\alpha\beta} \cdot \int \frac{d^3p\cdot c\hbar}{(2\pi\hbar)^3\,2E(\vec{p})} \begin{Bmatrix} e^{-\mathrm{i}\underline{p}(\underline{x}-\underline{y})/\hbar} & \text{for } x^0 > y_o \\ e^{\mathrm{i}\underline{p}(\underline{x}-\underline{y})/\hbar} & \text{for } \mathrm{y}^0 > x^0 \end{Bmatrix} \tag{3.4.21}$$

Introducing $\underline{q} = \underline{p} / \hbar$, $\left(q^0 = p^0 / \hbar = \frac{E(\vec{p})}{c\hbar} = \frac{\sqrt{c^2 \vec{p}^2 + m^2 c^4}}{c\hbar} \right)$, one can see that expression $(3.4.21)$ contains the so-called **scalar Feynman propagator** $\Delta_F(\underline{x} - \underline{y})$. It reads

$$\begin{aligned} \mathrm{i}\Delta_F(\underline{x} - \underline{y}) &= \int \frac{d^3 p\, c\hbar}{(2\pi\hbar)^3\, 2E(\vec{p})} \begin{Bmatrix} e^{-\mathrm{i}\underline{p}(\underline{x}-\underline{y})/\hbar} & \text{for } \mathrm{x}^0 > y^o \\ e^{\mathrm{i}\underline{p}(\underline{x}-\underline{y})/\hbar} & \text{for } \mathrm{y}^0 > x^0 \end{Bmatrix} \\ &= \int \frac{d^3 q}{(2\pi)^3\, 2q^0} \begin{Bmatrix} e^{-\mathrm{i}\underline{q}(\underline{x}-\underline{y})} & \text{for } \mathrm{x}^0 > y^o \\ e^{\mathrm{i}\underline{q}(\underline{x}-\underline{y})} & \text{for } \mathrm{y}^0 > x^0 \end{Bmatrix}. \end{aligned} \tag{3.4.22}$$

Expressions $(3.4.21)$ and $(3.4.22)$ reveal

$$\mathrm{i}S_{F,\alpha\beta}(\underline{x} - \underline{y}) = \left(\mathrm{i}\underline{\underline{\nabla\!\!\!/}}_{\underline{x}} + \frac{mc\underline{\underline{1}}}{\hbar} \right)_{\alpha\beta} \mathrm{i}\Delta_F(\underline{x} - \underline{y}) \tag{3.4.22a}$$

For $x^0 > y^0$ the expression$(3.4.22)$ reads (see $(3.4.8)$)

$$\int \frac{d^3 q}{(2\pi)^3\, 2q^0} e^{\mathrm{i}\vec{\mathrm{q}}(\vec{x}-\vec{y})}\, e^{-\mathrm{i}q^0(x^0 - y^0)} \tag{3.4.23}$$

and for $y^0 > x^0$

$$\int \frac{d^3 q}{(2\pi)^3\, 2q^0} e^{-\mathrm{i}\vec{q}(\vec{x}-\vec{y})}\, e^{\mathrm{i}q^0(x^0 - y^0)}. \tag{3.4.24}$$

In the integrand of $(3.4.24)$ we replace $\vec{q}$ by $-\vec{q}$. Doing so the function is reflected on the origin but $q^0 = E(\vec{p}) / \hbar c$ and the integral remain unchanged, i.e. $(3.4.22)$ becomes

$$\mathrm{i}\Delta_F(\underline{x} - \underline{y}) = \int \frac{d^3 q}{(2\pi)^3\, 2q^0}\, e^{\mathrm{i}\vec{q}(\vec{x}-\vec{y})} \begin{Bmatrix} e^{-\mathrm{i}q^0(x^0 - y^0)} & \text{for } x^0 > y^0 \\ e^{\mathrm{i}q^0(x^0 - y^0)} & \text{for } \mathrm{y}^0 > x^0 \end{Bmatrix}. \tag{3.4.25}$$

Because of

$$\left[e^{-\mathrm{i}q^0(x^0 - y^0)} \right]_{x^0 > y^0} = e^{-\mathrm{i}q^0 |x^0 - y^0|} = \left[e^{\mathrm{i}q^0(x^0 - y^0)} \right]_{y^0 > x^0}$$

it is obvious that the alternative expressions in $(3.4.25)$ can be replaced by one term containing $|x^0 - y^0|$. However, we choose another way which results in a four dimensional Lorenz invariant expression.

In order to transform the last part of $(3.4.25)$ in a closed form we look into the following line integral

$$I_C = \int_{C_F} \frac{dq'}{2\pi} \frac{e^{-iq'\left(x^0-y^0\right)}}{\left(q'-q^0\right)\left(q'+q^0\right)} \tag{3.4.26}$$

where q' is a complex variable and q^0 is real as definded below in $(3.4.21)$. The integration path C_F in the complex plane will be discussed. The integrand has singularities at $q' = q^0$ and at $q' = -q^0$. The complex coordinates relative to the singularity positions are

$$z = q' - q^0 \text{ and } z = q' + q^0 \text{ respectively.} \tag{3.4.27}$$

We write the integrand of $(3.4.26)$ using z like this

$$\frac{1}{2\pi} \frac{e^{-i\left(z\pm q^0\right)\left(x^0-y^0\right)}}{z\left(z\pm 2q^0\right)}. \tag{3.4.28}$$

The so-called residuum "Res" is the expansion coefficient of the term containing $\frac{1}{z}$ near a singularity, i.e.

$$\begin{aligned} \text{Res}\big|_{q'=q^0} &\simeq \frac{1}{2\pi} \frac{e^{-iq^0\left(x^0-y^0\right)}}{2q^0} \text{ and} \\ \text{Res}\big|_{q'=-q^0} &\simeq \frac{1}{2\pi} \frac{e^{iq^0\left(x^0-y^0\right)}}{-2q^0}. \end{aligned} \tag{3.4.29}$$

Cauchy's theorem of residues states for a closed integration contour around the singularity $q' = q^0$ in anti-clockwise sense

$$\oint_{\text{around } q^0} dq' \frac{1}{2\pi} \frac{e^{-iq'\left(x^0-y^0\right)}}{\left(q'\right)^2 - \left(q^0\right)^2} = 2\pi i \text{ Res}\big|_{q'=q^0} = i \frac{e^{-iq^0\left(x^0-y^0\right)}}{2q^0} \tag{3.4.30}$$

and for the other singularity we find

$$\oint_{\text{around } -q^0} dq' \frac{1}{2\pi} \frac{e^{-iq'\left(x^0-y^0\right)}}{\left(q'\right)^2 - \left(q^0\right)^2} = 2\pi i \text{ Res}\big|_{q'=-q^0} = i \frac{e^{iq^0\left(x^0-y^0\right)}}{-2q^0}. \tag{3.4.31}$$

If we choose Feynman's integration path C_F according to figure 3.4.1 and close it in the far negative imaginary part of the complex plane, the sense of the contour C_F around the point q^0 is clockwise (which changes the sign) and $(3.4.30)$ becomes

$$\oint_{C_F, -i\infty} dq' \frac{1}{2\pi} \frac{e^{-iq'\left(x^0-y^0\right)}}{\left(q'\right)^2 - \left(q^0\right)^2} = -i \frac{e^{-iq^0\left(x^0-y^0\right)}}{2q^0}. \tag{3.4.32}$$

Starting with C_F the contour goes around the point $-q^0$ anti-clockwise if we go back in the far positive imaginary region. Therefore, of $(3.4.31)$ remains

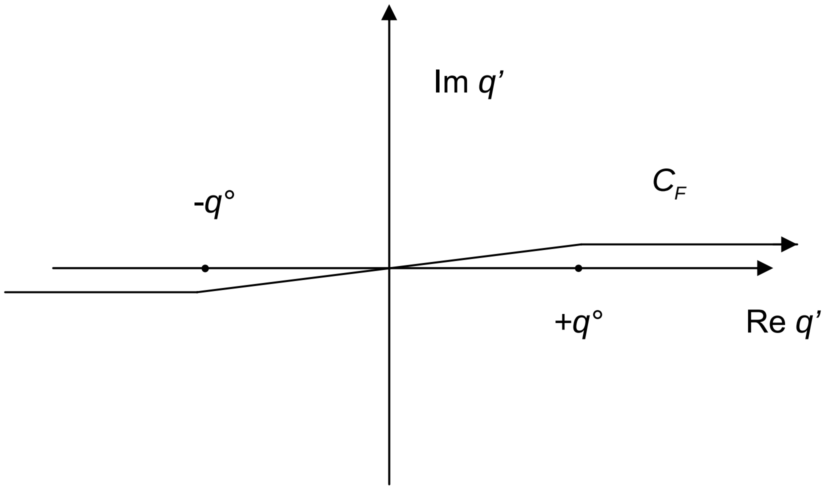

Figure 3.4.1 Integraton path for the scalar Feynman propagator

$$\oint_{C_F,+\mathrm{i}\infty} dq' \frac{1}{2\pi} \frac{e^{-\mathrm{i}q'\left(x^0-y^0\right)}}{\left(q'\right)^2-\left(q^0\right)^2} = -\mathrm{i}\frac{e^{\mathrm{i}q^0\left(x^0-y^0\right)}}{2q^0}. \tag{3.4.33}$$

Because far remote segments of the contour don't contribute to the line integrals, the left hand sides of $(3.4.32)$ and $(3.4.33)$ are identical and $(3.4.25)$ can be written as

$$\mathrm{i}\Delta_F\left(\underline{x}-\underline{y}\right) = \int \frac{d^3q}{(2\pi)^3} e^{\mathrm{i}\vec{q}(\vec{x}-\vec{y})}\, \mathrm{i}\int_{C_F} \frac{dq'}{2\pi} \frac{e^{-\mathrm{i}q'\left(x^0-y^0\right)}}{\left(q'\right)^2-\left(q^0\right)^2}\Bigg|_{\text{all } \left(x^0-y^0\right)} \tag{3.4.34}$$

We choose the integration path C_F infinitesimally near under the singularity $q' = -q^0$ and infinitesimally near over $q' = +q^0$. The same result as in the integral I_C, $(3.4.26)$, can be achieved if one integrates along the real q'-axis instead of the contour C_F and avoids hitting the singularities by shifting them by an infinitesimal amount is in the complex plane. I.e. we replace $q^0 \to q^0 - \mathrm{i}s$ and $-q^0 \to -q^0 + \mathrm{i}s$ and set q' real. Therefore

$$\left(q'\right)^2-\left(q^0\right)^2 \to \left(q'\right)^2-\left(q^0-\mathrm{i}s\right)^2 = \left(q'\right)^2-\left(q^0\right)^2+2\mathrm{i}q^0s+s^2$$
$$\simeq \left(q'\right)^2-\left(q^0\right)^2+\mathrm{i}\varepsilon \quad \text{with } 2q^0s=\varepsilon \text{ and } s^2\simeq 0.$$

Then $(3.4.34)$ reads

$$\Delta_F\left(\underline{x}-\underline{y}\right) \simeq \int \frac{d^4q}{(2\pi)^4} \frac{e^{-\mathrm{i}\underline{q}\left(\underline{x}-\underline{y}\right)}}{\left(q'\right)^2-\left(q^0\right)^2+\mathrm{i}\varepsilon} \tag{3.4.35}$$

with $\underline{q} = \left(q', \vec{q}\right)$ like $(3.4.8)$.

As in $(3.4.8)$ we define the scalar product of the four-dimensional vectors or spinors $\underline{a} = (a^0, \vec{a})$ with $\underline{b} = (b^0, \vec{b})$ as follows

$$\underline{a}\underline{b} = a^0 b^0 - \vec{a} \cdot \vec{b}$$
$$\text{and consequently } (\underline{a})^2 = (a^0)^2 - (\vec{a})^2 . \qquad (3.4.35a)$$

With $q^0 = \frac{p^0}{\hbar} = \frac{E(\vec{p})}{c\hbar} = \frac{\sqrt{c^2\vec{p}^2 + m^2c^4}}{c\hbar} = \frac{\sqrt{c^2\hbar^2\vec{q}^2 + m^2c^4}}{c\hbar} = \sqrt{\vec{q}^2 + \frac{m^2c^2}{\hbar^2}}$

we get

$$\begin{aligned} \Delta_F(\underline{x} - \underline{y}) &= \int \frac{d^4q}{(2\pi)^4} \frac{e^{-i\underline{q}(\underline{x}-\underline{y})}}{(q')^2 - (\vec{q})^2 - \left(\frac{mc}{\hbar}\right)^2 + i\varepsilon} \\ &= \int \frac{d^4q}{(2\pi)^4} \frac{e^{-i\underline{q}(\underline{x}-\underline{y})}}{(\underline{q})^2 - \left(\frac{mc}{\hbar}\right)^2 + i\varepsilon}, \end{aligned} \qquad (3.4.36)$$

where we have used $\underline{q} = (q', \vec{q})$ and $(\underline{q})^2 = (q')^2 - (\vec{q})^2$.

Equation $(3.4.36)$ conveys that

$$\Delta_F(\underline{q}) \equiv \frac{1}{(\underline{q})^2 - \left(\frac{mc}{\hbar}\right)^2 + i\varepsilon} \qquad (3.4.37)$$

is the four-dimensional Fourier transform of the scalar Feynman propagator $\Delta_F(\underline{x} - \underline{y})$. We write the Feynman propagator $S_{F,\alpha\beta}(\underline{x} - \underline{y})$, $(3.4.22a)$, once more using $(3.4.20)$

$$\begin{aligned} S_{F,\alpha\beta}(\underline{x} - \underline{y}) &= \left(i\underline{\underline{\nabla\!\!\!/}}_{\underline{x}} + \frac{mc\underline{\underline{1}}}{\hbar}\right)_{\alpha\beta} \Delta_F(\underline{x} - \underline{y}) \\ &= \left(i\underline{\underline{\nabla\!\!\!/}}_{\underline{x}} + \frac{mc\underline{\underline{1}}}{\hbar}\right)_{\alpha\beta} \int \frac{d^4q}{(2\pi)^4} \frac{e^{-i\underline{q}(\underline{x}-\underline{y})}}{(\underline{q})^2 - \left(\frac{mc}{\hbar}\right)^2 + i\varepsilon} \\ &= \int \frac{d^4q}{(2\pi)^4} \frac{e^{-i\underline{q}(\underline{x}-\underline{y})} \left(\underline{\underline{q\!\!\!/}} + \frac{mc}{\hbar}\underline{\underline{1}}\right)_{\alpha\beta}}{(\underline{q})^2 - \left(\frac{mc}{\hbar}\right)^2 + i\varepsilon}. \end{aligned} \qquad (3.4.38)$$

The Feynman propagator $S_{F,\alpha\beta}\left(\underline{x}-\underline{y}\right)$ and the scalar Feynman propagator $\Delta_F\left(\underline{x}-\underline{y}\right)$ will be seen to emerge naturally in the perturbation series for interacting field theories (sections 5.4 and 5.7).

4 The electromagnetic field

4.1 The Maxwell equations

We put together basic electromagnetic relations, which we write in the SI-system of units (cf. Pfeifer, 2000, p.7). The electric field strength $\vec{E}$ and the magnetic field strength $\vec{B}$ can be calculated starting from the scalar potential Φ and the vector potential $\vec{A}$ like this

$$\vec{E} = -\text{grad}\,\Phi - \frac{d\vec{A}}{dt}, \tag{4.1.1}$$

$$\vec{B} = \text{curl}\,\vec{A} \equiv \vec{\nabla} \times \vec{A}, \tag{4.1.2}$$

$$\text{for example } B_x = \frac{dA_z}{dy} - \frac{dA_y}{dz}.$$

The potentials Φ and $\vec{A}$ obey differential equations, which contain the electric charge density ρ and the electric current density $\vec{j}$. The equations read

$$\frac{1}{c^2}\frac{d^2\Phi}{dt^2} - \nabla^2\Phi = \frac{\rho}{\varepsilon_0}, \tag{4.1.3}$$

$$\frac{1}{c^2}\frac{d^2\vec{A}}{dt^2} - \nabla^2\vec{A} = \mu_0\vec{j}\,. \tag{4.1.4}$$

The quantities $\mu_o = 4\pi 10^{-7}\,\text{VsA}^{-1}\text{m}^{-1}$ and $\varepsilon_0 = 8.85418782\cdot 10^{-12}\,\text{AsV}^{-1}\text{m}^{-1}$ are the magnetic field constant and the electric field constant respectively. Consequently, the quantities Φ / c and $\vec{A}$ have the dimension $[\text{momentum/charge}]$. An additional constraint can be chosen. We take the Lorenz gage

$$\text{div}\vec{A} + \frac{1}{c^2}\frac{d\Phi}{dt} = 0. \tag{4.1.5}$$

In the framework of special relativity it is natural to introduce the contravariant four-vectors

$$\underline{A} = \left(A^0, A^{(1)}, A^{(2)}, A^{(3)}\right) \equiv \left(\frac{\Phi}{c}, A_x, A_y, A_z\right) \tag{4.1.6}$$

and

$$\underline{J} = \left(J^0, J^{(1)}, J^{(2)}, J^{(3)}\right) \equiv \left(c\rho, j_x, j_y, j_z\right). \tag{4.1.7}$$

Making use of $(3.4.3), (4.1.6)$ and $(4.1.7)$ we rewrite $(4.1.3)$

$$\frac{d^2A^0}{\left(dx^0\right)^2}-\sum_{i=1}^{3}\frac{d^2A^0}{\left(dx^{(i)}\right)^2}=\frac{\rho}{c\varepsilon_0}=\frac{J^0}{c^2\varepsilon_0}=\mu_0 J^0\,, \tag{4.1.8}$$

where we have applied the basic relation

$$\varepsilon_0\mu_0=\frac{1}{c^2}\,. \tag{4.1.9}$$

Of course, using $(4.1.6)$ and $(4.1.7)$ the equation $(4.1.4)$ combined with $(4.1.8)$ becomes

$$\frac{d^2A^{(\mu)}}{\left(dx^0\right)^2}-\sum_{i=1}^{3}\frac{d^2A^{(\mu)}}{\left(dx^{(i)}\right)^2}=\mu_0 J^{(\mu)} \qquad \mu=0,1,2,3\,. \tag{4.1.10}$$

The Lorenz gage, $(4.1.5)$, reads now

$$\sum_{\mu=0}^{3}\frac{dA^{(\mu)}}{dx^{(\mu)}}=0\,. \tag{4.1.11}$$

4.2 The Lagrangian and the Hamiltonian of the Maxwell field

First, we introduce the relativistic notations of covariant four-vectors

$$\left(A_0,A_1,A_2,A_3\right)=\left(A^0,-A^{(1)},-A^{(2)},-A^{(3)}\right) \quad \text{or } A_\mu=A^{(\mu)}g_{\mu\mu} \tag{4.2.1}$$

with the following elements of the metric tensor $\underline{\underline{g}}$

$$g_{00}=-g_{11}=-g_{22}=-g_{33}=1 \;\text{ and }\; g_{\lambda\lambda'}=0 \;\text{ for }\; \lambda\neq\lambda'\,, \tag{4.2.2}$$

and we define

$$J_\mu=J^{(\mu)}g_{\mu\mu}\,. \tag{4.2.3}$$

Analogously the four-coordinate $x_\mu=x^{(\mu)}g_{\mu\mu}$ is used. We claim that the Lagrange density of the electromagnetic field can be chosen as

$$\begin{aligned}\mathcal{L}&=\frac{-1}{2\mu_0}\sum_{\mu,\nu=0}^{3}\frac{dA_\nu}{dx^{(\mu)}}\frac{dA^{(\nu)}}{dx_\mu}-\sum_{\rho=0}^{3}J_\rho A^{(\rho)}\\ &=\frac{-1}{2\mu_0}\sum_{\mu,\nu=0}^{3}g_{\mu\mu}g_{\nu\nu}\left(\frac{dA^{(\nu)}}{dx^{(\mu)}}\right)^2-\sum_{\rho=0}^{3}J_\rho A^{(\rho)}.\end{aligned} \tag{4.2.4}$$

In order to check this ansatz we apply the Euler-Lagrange equation, $(1.3.16)$. It reads

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{A}^{(\nu)}}+\sum_{i=1}^{3}\frac{d}{dx^{(i)}}\frac{\partial\mathcal{L}}{\partial\left(\frac{dA^{(\nu)}}{dx^{(i)}}\right)}-\frac{\partial\mathcal{L}}{\partial A^{(\nu)}}=0 \quad \text{or}$$

$$\sum_{\mu=0}^{3}\frac{d}{dx^{(\mu)}}\frac{\partial\mathcal{L}}{\partial\left(\frac{dA^{(\nu)}}{dx^{(\mu)}}\right)}-\frac{\partial\mathcal{L}}{\partial A^{(\nu)}}=0\ . \tag{4.2.5}$$

Here, the following derivations appear

$$\frac{d}{dx^{(\mu)}}\frac{\partial\mathcal{L}}{\partial\left(\frac{dA^{(\nu)}}{dx^{(\mu)}}\right)}=-\frac{1}{\mu_0}g_{\mu\mu}g_{\nu\nu}\frac{d}{dx^{(\mu)}}\frac{dA^{(\nu)}}{dx^{(\mu)}}=-\frac{1}{\mu_0}g_{\mu\mu}g_{\nu\nu}\frac{d^2A^{(\nu)}}{\left(dx^{(\mu)}\right)^2} \tag{4.2.6}$$

and

$$-\frac{\partial\mathcal{L}}{\partial A^{(\nu)}}=J_\nu=g_{\nu\nu}J^{(\nu)}. \tag{4.2.7}$$

We insert $(4.2.6)$ and $(4.2.7)$ in $(4.2.5)$ and obtain

$$-\frac{1}{\mu_0}\sum_{\mu=0}^{3}g_{\mu\mu}\frac{d^2A^{(\mu)}}{\left(dx^{(\mu)}\right)^2}+J^{(\nu)}=0 \quad \text{or}$$

$$\frac{d^2A^{(\nu)}}{\left(dx^0\right)^2}-\sum_{i=1}^{3}\frac{d^2A^{(\nu)}}{\left(dx^{(i)}\right)^2}=\mu_0J^{(\nu)} \tag{4.2.8}$$

in accordance with $(4.1.10)$. Thus, the expression $(4.2.4)$ is corroborated.

Due to $(1.4.13)$ the Hamilton density of the electromagnetic field in vacuum $(\underline{J}=0)$ reads

$$\begin{aligned}
\mathcal{H}&=\sum_{\nu=0}^{3}\frac{\partial\mathcal{L}}{\partial\left(\frac{dA^{(\nu)}}{dx^0}\right)}\cdot\frac{dA^{(\nu)}}{dx^0}-\mathcal{L}\\
&=-\frac{1}{2\mu_0}\sum_{\nu=0}^{3}g_{\nu\nu}2\left(\frac{dA^{(\nu)}}{dx^0}\right)^2+\frac{1}{2\mu_0}\sum_{\mu,\nu=0}^{3}g_{\mu\mu}g_{\nu\nu}\left(\frac{dA^{(\nu)}}{dx^{(\mu)}}\right)^2\\
&=\frac{1}{\mu_0}\sum_{\nu=0}^{3}g_{\nu\nu}\left[-\left(\frac{dA^{(\nu)}}{dx^0}\right)^2+\frac{1}{2}\left(\frac{dA^{(\nu)}}{dx^0}\right)^2-\frac{1}{2}\sum_{m=1}^{3}\left(\frac{dA^{(\nu)}}{dx^{(m)}}\right)^2\right]\\
&=\frac{1}{\mu_0}\sum_{\nu=0}^{3}g_{\nu\nu}\frac{1}{2}\left[-\left(\frac{dA^{(\nu)}}{dx^0}\right)^2-\left(\vec{\nabla}A^{(\nu)}\right)^2\right]\\
&=\frac{1}{2\mu_0}\left[-\left(\left(\frac{dA^0}{dx^0}\right)^2+\left(\vec{\nabla}A^0\right)^2\right)+\sum_{n=1}^{3}\left(\left(\frac{dA^{(n)}}{dx^0}\right)^2+\left(\vec{\nabla}A^{(n)}\right)^2\right)\right]
\end{aligned} \tag{4.2.9}$$

This expression is not positive definite, however, it can be shown (Greiner, Reinhardt, 1996, p. 174) that the total Hamiltonian $\int d^3x\mathcal{H}(\underline{x})$ agrees with the familiar form

$$\int d^3x\ \mathcal{H}(\underline{x}) = \int d^3x \frac{1}{2}\left(\vec{E}^2 + \vec{B}^2\right). \tag{4.2.10}$$

4.3 Coupled Maxwell and Dirac fields

The Lagrange density of Fermions, $(3.1.7)$, and the Lagrangian of photons, $(4.2.4)$, are linear combinations of quadratic expressions of the affiliated wave functions or their derivations. Therefore, it is obvious that the Lagrange density of the coupled fields is just the sum of both Lagrangians like this

$$\mathcal{L} = \mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\text{Maxwell}} \tag{4.3.1}$$

By means of $(3.1.7)$ and $(4.2.4)$ it turns into

$$\mathcal{L} = \underline{\psi}^\dagger\left(\mathrm{i}\hbar\dot{\underline{\psi}} + \mathrm{i}\sum_{i=1}^{3}\hbar c\underline{\underline{\alpha}}_i\frac{d\underline{\psi}}{dx^i} - mc^2\underline{\underline{\beta}}\underline{\psi}\right) - \frac{1}{2\mu_0}\sum_{\mu,\nu=0}^{3}\frac{dA_\nu}{dx^{(\mu)}}\frac{dA^{(\nu)}}{dx_\mu} - \sum_{\rho=0}^{3}J_\rho A^{(\rho)}\ . \tag{4.3.2}$$

$\mathcal{L}$ has the dimension [energy/volume]. With the aid of $(3.1.4)$ we join the expressions for the current density, $(3.1.16)$, and for the charge density, $(3.1.17)$, together this way (using $(4.1.7)$ and $(4.2.3)$)

$$J_\mu(\vec{x},t) = g_{\mu\mu}J^{(\mu)}(\vec{x},t) = g_{\mu\mu}ec\underline{\psi}^\dagger\underline{\underline{\alpha}}_\mu\underline{\psi}(\vec{x},t), \qquad \mu = 0,1,2,3\ . \tag{4.3.3}$$

This expression is inserted in $(4.3.2)$, through which the Maxwell field and the Dirac field are coupled together. The Euler-Lagrange equation $(3.1.9)$ (with the hermitean conjugate counterpart) and $(4.2.5)$ can be applied separately. If $(3.1.9)$ acts on the Lagrangian $(4.3.2)$ including $(4.3.3)$, the result is

$$\begin{aligned}&\mathrm{i}\hbar\dot{\psi}_k + \mathrm{i}\hbar c\sum_{i=1}^{3}\underline{\underline{\alpha}}_i\frac{d\psi_k}{dx^{(i)}} - mc^2\left(\underline{\underline{\beta}}\underline{\psi}\right)_k - ec\sum_{\mu=0}^{3}\left(\underline{\underline{\alpha}}_\mu\underline{\psi}\right)_k A^{(\mu)} = 0 \quad (k = 1,\cdots,4)\\ &\text{or}\quad \mathrm{i}\hbar c\sum_{\mu=0}^{3}\underline{\underline{\alpha}}_\mu\frac{d\underline{\psi}}{dx^{(\mu)}} - mc^2\underline{\underline{\beta}}\underline{\psi} - ec\sum_{\mu=0}^{3}\underline{\underline{\alpha}}_\mu\underline{\psi}A^{(\mu)}g_{\mu\mu} = 0.\end{aligned} \tag{4.3.4}$$

Replacing $\psi_k^\dagger$ by ψ_k in $(3.1.9)$ we obtain analogously

$$\mathrm{i}\hbar c\sum_{\mu=0}^{3}\frac{d\underline{\psi}^\dagger}{dx^{(\mu)}}\underline{\underline{\alpha}}_\mu + mc^2\underline{\psi}^\dagger\underline{\underline{\beta}} + ec\sum_{\mu=0}^{3}\underline{\psi}^\dagger\underline{\underline{\alpha}}_\mu A_\mu g_{\mu\mu} = 0 \tag{4.3.5}$$

Finally we apply the Euler-Lagrange equation $(4.2.5)$ to the Lagrangian $(4.3.2)$ including $(4.3.3)$ and obtain according to $(4.2.8)$ and $(4.3.3)$

$$\frac{1}{\mu_0}\left(\frac{d^2A^{(\mu)}}{\left(dx^0\right)^2}-\sum_{i=1}^{3}\frac{d^2A^{(\mu)}}{\left(dx^{(i)}\right)^2}\right)=J^{(\mu)}=ec\underline{\psi}^\dagger\underline{\underline{\alpha}}_\mu\underline{\psi}\ . \tag{4.3.6}$$

In section 5.5 we will need the Lagrangian of the quantized interacting fields. The derived equations $(4.3.4)$ up to $(4.3.6)$ represent a set of coupled equations of motion for the fields $\underline{\psi},\underline{\psi}^\dagger$ and $\underline{A}$.

4.4 Plane wave expansion of the vector field

According to $(4.1.10)$ or $(4.3.6)$ the electromagnetic field in vacuum obeys the following equation

$$\frac{d^2A_\mu}{\left(dx^0\right)^2}-\sum_{i=1}^{3}\frac{d^2A_\mu}{\left(dx^{(i)}\right)^2}=0 \tag{4.4.1}$$

The vector field component

$$A_\mu\left(\vec{k},\underline{x}\right)=N_\mu\left(\vec{k}\right)e^{\mathrm{i}\left(\vec{k}\vec{x}-k^0x^0\right)/\hbar} \tag{4.4.2}$$

meets this relation. It represents a plane wave propagating in the $\vec{k}$-direction as the Dirac wave in $(3.2.1)$. The dimension of A_μ and N_μ is [momentum/charge]. Inserting $(4.4.2)$ in $(4.4.1)$ results in

$$\left(k^0\right)^2=\vec{k}^2 \tag{4.4.2a}$$

The dimension of $k^{(\nu)}$ is [momentum]. Using $(3.4.3)$ we write also

$$\begin{aligned}&A_\mu\left(\vec{k},\underline{x}\right)=N_\mu\left(\vec{k}\right)e^{\mathrm{i}\left(\vec{k}\vec{x}-\omega_k t\right)/\hbar}\\ &\text{with }\ \omega_k=ck^0=c\left|\vec{k}\right|.\end{aligned} \tag{4.4.3}$$

The dimension of ω_k is [energy].

Equations $(4.1.2)$ and $(4.1.6)$ show that the components $A^{(1)},A^2$ and $A^{(3)}$ fix the magnetic field strength $\vec{B}$. The electric field strength $\vec{E}$ is determined by $A^0,\dot{A}^{(1)},\dot{A}^{(2)}$ and $\dot{A}^{(3)}$. It is well-known that the $\vec{B}'$s and the $\vec{E}'$s of free electromagnetic waves are orthogonal to each other and to the propagation vector $\vec{k}$. To take this property into account we split up the vector field A_μ into components like this

$$A_{\mu\lambda}\left(\vec{k},\underline{x}\right)=N_\mu\left(\vec{k}\right)e^{\mathrm{i}\left(\vec{k}\vec{x}-\omega_k t\right)/\hbar}\varepsilon_{\mu,\lambda}\left(\vec{k}\right)\qquad \lambda=0,1,2,3\,. \tag{4.4.4}$$

Due to the electrodynamics the quantities $\varepsilon_{\mu\lambda}\left(\vec{k}\right)$ are four-dimensional, orthonormalized unit vectors affiliated to $A_{\mu\lambda}\left(\vec{k},\underline{x}\right)$ and oriented relative to $\vec{k}$ .They meet the four-dimensional orthogonality relation

$$\varepsilon_{0,\lambda}\left(\vec{k}\right)\varepsilon_{0,\lambda'}\left(\vec{k}\right)-\sum_{m=1}^{3}\varepsilon_{m\lambda}\left(\vec{k}\right)\varepsilon_{m\lambda'}\left(\vec{k}\right)=g_{\lambda\lambda'} \tag{4.4.5}$$

containing the metric tensor $(4.2.2)$. In order to specify the four-vectors $\underline{\varepsilon}_{\lambda}\left(\vec{k}\right)$ we start with a trihedral of three-vectors $\vec{\varepsilon}_{\lambda=1}\left(\vec{k}\right),\vec{\varepsilon}_{\lambda=2}\left(\vec{k}\right)$ and $\vec{\varepsilon}_{\lambda=3}\left(\vec{k}\right)$. We put

$$\vec{\varepsilon}_{\lambda=3}\left(\vec{k}\right)=\frac{\vec{k}}{\left|\vec{k}\right|} \tag{4.4.6}$$

and choose the other three-vectors to be orthogonal to $\vec{k}$

$$\vec{k}\cdot\vec{\varepsilon}_{\lambda=1}\left(\vec{k}\right)=\vec{k}\cdot\vec{\varepsilon}_{\lambda=2}\left(\vec{k}\right)=0\,. \tag{4.4.7}$$

Furthermore,

$$\begin{gathered}\left(\vec{\varepsilon}_{\lambda=1}\left(\vec{k}\right)\right)^{2}=\left(\vec{\varepsilon}_{\lambda=2}\left(\vec{k}\right)\right)^{2}=1 \quad \text{and} \\ \vec{\varepsilon}_{\lambda=1}\left(\vec{k}\right)\cdot\vec{\varepsilon}_{\lambda=2}\left(\vec{k}\right)=0.\end{gathered} \tag{4.4.8}$$

The four-vectors $\underline{\varepsilon}_{\lambda}\left(\vec{k}\right)\left(\lambda=0,\cdots,3\right)$ are fixed as follows

$$\underline{\varepsilon}_{l}\left(\vec{k}\right)=\left(0,\vec{\varepsilon}_{l}\left(\vec{k}\right)\right) \quad \text{with } l=1,2,3, \tag{4.4.9}$$

$$\underline{\varepsilon}_{\lambda=0}\left(\vec{k}\right)=\left(1,0,0,0\right). \tag{4.4.10}$$

We inspect whether these expressions fulfil the relation $(4.4.5)$

for $\lambda=\lambda'=0$: $\left(\varepsilon_{0,\lambda=0}\left(\vec{k}\right)\right)^{2}=1,\quad \sum_{m=1}^{3}\left(\varepsilon_{m,\lambda=0}\left(\vec{k}\right)\right)^{2}=0$ for $m>0$,

for $\lambda=\lambda'>0$: $\left(\varepsilon_{0,\lambda}\left(\vec{k}\right)\right)^{2}=0,\quad -\sum_{m=1}^{3}\left(\varepsilon_{m,\lambda}\left(\vec{k}\right)\right)^{2}=-1,$

for $\lambda=0,\lambda'>0$: $\varepsilon_{0,\lambda=0}\left(\vec{k}\right)\varepsilon_{0,\lambda'}\left(\vec{k}\right)=0,\quad \sum_{m=1}^{3}\varepsilon_{m,\lambda=0}\left(\vec{k}\right)\varepsilon_{m,\lambda'}\left(\vec{k}\right)=0,$

for 0<$\lambda\neq\lambda'>0$: $\varepsilon_{0,\lambda}\left(\vec{k}\right)\varepsilon_{0,\lambda'}\left(\vec{k}\right)=0,\quad \sum_{m=1}^{3}\varepsilon_{m,\lambda}\left(\vec{k}\right)\varepsilon_{m,\lambda'}\left(\vec{k}\right)=0.$

Obviously the expressions $(4.4.9)$ and $(4.4.10)$ fulfil the normalization relation $(4.4.5)$. Of course, every $A_{\mu\lambda}\left(\vec{k},\underline{x}\right)$, $(4.4.4)$, is a solution of the equation $(4.4.1)$.

As mentioned above the field strengths $\vec{E}$ and $\vec{B}$ of a free electromagnetic wave are orthogonal to $\vec{k}$ and only transverse photons are observed in experiments. This is true if the following λ-components vanish

$$\underline{A}_{\lambda=0}\left(\vec{k},\underline{x}\right)=\underline{A}_{\lambda=3}\left(\vec{k},\underline{x}\right)=\underline{0}\,. \tag{4.4.11}$$

However, in the theory of the interaction of quantized fields all λ-components of $\underline{A}$ play a role (cf. sections 4.7, 5.5 and 5.7).

4.5 Canonical quantization of the photon field

For the photon field one can formulate commutation rules analogously to sections 2.1 or 3.3. Due to $(1.4.14)$ the canonically conjugate field component reads (see Greiner, Reinhardt, 1996,p.158 at the bottom)

$$\pi^{(\nu)}\left(\vec{x},t\right)=\frac{\partial\mathcal{L}}{\partial\left(\dfrac{dA_\nu}{dt}\right)} \tag{4.5.1}$$

or in relativistic coordinates, $(3.4.3)$,

$$\pi^{(\nu)}\left(\underline{x}\right)=\frac{\partial\mathcal{L}}{c\partial\left(\dfrac{dA_\nu}{dx^0}\right)}\,. \tag{4.5.2}$$

With the Lagrangian $(4.2.4)$ we obtain

$$\begin{aligned}\pi^{(\nu)}\left(\underline{x}\right)&=\partial\left(-\frac{1}{2\mu_0}\sum_{\mu,\lambda=0}^{3}g_{\mu\mu}g_{\lambda\lambda}\left(\frac{dA_\lambda}{dx^{(\mu)}}\right)^2\right)/\,\partial\left(\frac{dA_\nu}{dx^0}\right)c\\&=\frac{-g_{\nu\nu}}{\mu_0 c}\frac{dA_\nu}{dx^0}=\frac{-1}{\mu_0 c}\frac{dA^{(\nu)}}{dx^0}=-\frac{\dot{A}^{(\nu)}}{\mu_0 c^2}=-\dot{A}^{(\nu)}\varepsilon_0\,,\end{aligned} \tag{4.5.3}$$

where $(4.1.9)$ has been used.

For the second quantization, as in section 2.1, we replace fields by field operators

$$\begin{aligned}A^{(\mu)}\left(\underline{x}\right)&\rightarrow\boldsymbol{A}^{(\mu)}\left(\underline{x}\right),\\\pi^{(\nu)}\left(\underline{x}\right)&\rightarrow\boldsymbol{\pi}^{(\nu)}\left(\underline{x}\right).\end{aligned} \tag{4.5.4}$$

We postulate the following equal-time commutation relations

$$\left[\boldsymbol{A}^{(\mu)}\left(\vec{x},t\right),\boldsymbol{\pi}^{(\nu)}\left(\vec{x}',t\right)\right]_-=\mathrm{i}g_{\mu\nu}\hbar\delta^3\left(\vec{x}-\vec{x}'\right). \tag{4.5.5}$$

In contrast to $(2.1.9)$ here the metric tensor, $(4.2.2)$, appears. Due to $(4.1.4)$ and $(4.5.3)$ the dimensions of $\underline{\boldsymbol{A}}$ and $\underline{\boldsymbol{\pi}}$ are [momentum/charge] and [charge/area] respectively. Therefore, in $(4.5.5)$ the dimension balance is correct. By means of $(4.5.3)$ we have also

$$\varepsilon_0 \left[\boldsymbol{A}^{(\mu)}(\vec{x},t), \dot{\boldsymbol{A}}^{(\nu)}(\dot{\vec{x}},t) \right]_- = -\mathrm{i} g_{\mu\nu} \hbar \delta^3 (\vec{x} - \vec{x}'). \tag{4.5.6}$$

The most general solution of the wave equation $(4.4.1)$ for photons is a linear combination of expressions $(4.4.4)$ and their conjugate complex form, which is integrated over $\vec{k}$

$$\begin{aligned} A^{(\mu)}(\underline{x}) = \int \frac{1}{\sqrt{\varepsilon_0 \hbar}} \frac{d^3 k}{\sqrt{2\omega_k (2\pi)^3}} \\ \cdot \sum_{\lambda=0}^{3} \left(a_\lambda(\vec{k}) \varepsilon_{\mu,\lambda}(\vec{k}) e^{\mathrm{i}(\vec{k}\vec{x} - \omega_k t)/\hbar} + \bar{a}_\lambda(\vec{k}) \varepsilon_{\mu,\lambda}(\vec{k}) e^{-\mathrm{i}(\vec{k}\vec{x} - \omega_k t)/\hbar} \right) \end{aligned} \tag{4.5.7}$$

with $\omega_k{}^2 = \vec{k}^2 c^2$ as per $(4.4.3)$. The normalizing constant $1/\sqrt{\varepsilon_0 \hbar 2\omega_k (2\pi)^3}$ is arbitrary, but taking into account $(4.5.11)$ it gives the quantity $\underline{A}$ the same dimension [momentum/charge] as in the inhomogeneous equations $(4.1.3), (4.1.4)$ and in $(4.5.5)$.

Similarly to $(3.3.1)$ the second quantization is performed by replacing $a_\lambda(\vec{k})$ and $\bar{a}_\lambda(\vec{k})$ by operators

$$\begin{aligned} a_\lambda(\vec{k}) &\to \boldsymbol{a}_\lambda(\vec{k}) \\ \bar{a}_\lambda(\vec{k}) &\to \boldsymbol{a}_\lambda^\dagger(\vec{k}). \end{aligned} \tag{4.5.8}$$

$$\begin{aligned} \text{I.e. } \boldsymbol{A}^{(\mu)}(\underline{x}) = \int \frac{1}{\sqrt{\varepsilon_0 \hbar}} \frac{d^3 k}{\sqrt{2\omega_k (2\pi)^3}} \\ \cdot \sum_{\lambda=0}^{3} \left(\boldsymbol{a}_\lambda(\vec{k}) \varepsilon_{\mu,\lambda}(\vec{k}) e^{\mathrm{i}(\vec{k}\vec{x} - \omega_k t)/\hbar} + \boldsymbol{a}_\lambda^\dagger(\vec{k}) \varepsilon_{\mu,\lambda}(\vec{k}) e^{-\mathrm{i}(\vec{k}\vec{x} - \omega_k t)/\hbar} \right). \end{aligned} \tag{4.5.9}$$

According to $(4.5.3)$ the canonically conjugate field operator $\boldsymbol{\pi}^{(\mu)} = -\dot{\boldsymbol{A}}^{(\mu)} \varepsilon_0$ reads

$$\begin{aligned} \boldsymbol{\pi}^{(\mu)}(\underline{x}) = \mathrm{i} \int \frac{1}{\sqrt{\varepsilon_0 \hbar}} \frac{d^3 k}{\sqrt{2\omega_k (2\pi)^3}} \frac{\omega_k \varepsilon_0}{\hbar} \\ \cdot \sum_{\lambda=0}^{3} \left(\boldsymbol{a}_\lambda(\vec{k}) \varepsilon_{\mu,\lambda}(\vec{k}) e^{\mathrm{i}(\vec{k}\vec{x} - \omega_k t)/\hbar} - \boldsymbol{a}_\lambda^\dagger(\vec{k}) \varepsilon_{\mu,\lambda}(\vec{k}) e^{-\mathrm{i}(\vec{k}\vec{x} - \omega_k t)/\hbar} \right). \end{aligned} \tag{4.5.10}$$

We claim that the operators $\boldsymbol{a}_\lambda(\vec{k})$ meet the following commutation rules

$$\left[\boldsymbol{a}_\lambda\left(\vec{k}\right),\boldsymbol{a}^\dagger_{\lambda'}\left(\vec{k}'\right)\right]_- = -g_{\lambda\lambda'}\delta^3\left(\vec{k}-\vec{k}'\right) \tag{4.5.11}$$

and
$$\left[\boldsymbol{a}_\lambda\left(\vec{k}\right),\boldsymbol{a}_{\lambda'}\left(\vec{k}'\right)\right]_- = \left[\boldsymbol{a}_\lambda^{\dagger}\left(\vec{k}\right),\boldsymbol{a}_{\lambda'}^{\dagger}\left(\vec{k}'\right)\right]_- = 0\,. \tag{4.5.12}$$

We verify $(4.5.5)$ using $(4.5.9)$ up to $(4.5.12)$

$$\begin{aligned}\left[\boldsymbol{A}^{(\mu)}(\vec{x},t),\boldsymbol{\pi}^{(\nu)}(\vec{x}',t)\right]_- &= \frac{\mathrm{i}}{\hbar}\int\frac{d^3k}{\sqrt{2\omega_k(2\pi)^3}}\int\frac{d^3k'}{\sqrt{2\omega_{k'}(2\pi)^3}}\frac{\omega_{k'}}{\hbar}\\ &\cdot\sum_{\lambda=0}^{3}\sum_{\lambda'=0}^{3}\Big(-\left[\boldsymbol{a}_\lambda\left(\vec{k}\right),\boldsymbol{a}^\dagger_{\lambda'}\left(\vec{k}'\right)\right]_-\varepsilon_{\mu,\lambda}\left(\vec{k}\right)\varepsilon_{\nu,\lambda'}\left(\vec{k}'\right)e^{\mathrm{i}\left(\vec{k}\vec{x}-\vec{k}'\vec{x}'\right)/\hbar}e^{-\mathrm{i}(\omega_k-\omega_{k'})t/\hbar}\\ &\quad+\left[\boldsymbol{a}^\dagger_\lambda\left(\vec{k}\right),\boldsymbol{a}_{\lambda'}\left(\vec{k}'\right)\right]_-\varepsilon_{\mu,\lambda}\left(\vec{k}\right)\varepsilon_{\nu,\lambda'}\left(\vec{k}'\right)e^{\mathrm{i}\left(-\vec{k}\vec{x}+\vec{k}'\cdot\vec{x}'\right)/\hbar}e^{\mathrm{i}(\omega_k-\omega_{k'})t/\hbar}\Big)\\ &= \frac{\mathrm{i}}{\hbar^2}\int\frac{d^3k}{2(2\pi)^3}\sum_{\lambda=0}^{3}g_{\lambda\lambda}\varepsilon_{\mu,\lambda}\left(\vec{k}\right)\varepsilon_{\nu,\lambda}\left(\vec{k}\right)\left(e^{\mathrm{i}(\vec{x}-\vec{x}')\vec{k}/\hbar}+e^{-\mathrm{i}(\vec{x}-\vec{x}')\vec{k}/\hbar}\right).\end{aligned} \tag{4.5.13}$$

We deal with the λ-sum in the last line of $(4.5.13)$ and assert

$$\sum_{\lambda=0}^{3}g_{\lambda\lambda}\varepsilon_{\mu,\lambda}\left(\vec{k}\right)\varepsilon_{\nu,\lambda}\left(\vec{k}\right) = g_{\mu\nu}\,. \tag{4.5.14}$$

Using $(4.4.9\ )$ and $(4.4.10)$ the left hand side of $(4.5.14)$ can be written as

$$\begin{pmatrix}1\\ \vec{0}\end{pmatrix}_\mu\begin{pmatrix}1\\ \vec{0}\end{pmatrix}_\nu-\sum_{\lambda=1}^{3}\begin{pmatrix}0\\ \vec{\varepsilon}_\lambda\left(\vec{k}\right)\end{pmatrix}_\mu\begin{pmatrix}0\\ \vec{\varepsilon}_\lambda\left(\vec{k}\right)\end{pmatrix}_\nu,$$

where the μ-component and the ν-component of the four-vectors are taken.

If $\mu=\nu=0$ the result is +1. If $\mu=0<\nu$ or $\mu>0=\nu$ the expression vanishes. For $\mu>0$ and $\nu>0$ it reads

$$-\sum_{\lambda=1}^{3}\varepsilon_{i,\lambda}\left(\vec{k}\right)\varepsilon_{j,\lambda}\left(\vec{k}\right),\qquad i,j,=1,2,3\,.$$

This is just the ordinary completeness relation for an orthogonal trihedral and amounts to $-\delta_{ij}$. Therefore, relation $(4.5.14)$ holds and using $(3.2.25)$ we have

$$\begin{aligned}&\left[\boldsymbol{A}^{(\mu)}(\vec{x},t),\boldsymbol{\pi}^{(\nu)}(\vec{x}',t)\right]\\ &= \frac{\mathrm{i}}{\hbar^2}\int\frac{d^3k}{2(2\pi)^3}g_{\mu\nu}\left(e^{\mathrm{i}(\vec{x}-\vec{x}')\vec{k}/\hbar}+e^{-\mathrm{i}(\vec{x}-\vec{x}')\vec{k}/\hbar}\right)\\ &= \frac{\mathrm{i}}{\hbar^2}\frac{1}{2(2\pi)^3}g_{\mu\nu}\hbar^3(2\pi)^3\,2\,\delta^3\left(\vec{x}-\vec{x}'\right) = \mathrm{i}g_{\mu\nu}\delta^3\left(\vec{x}-\vec{x}'\right)\hbar\end{aligned} \tag{4.5.15}$$

in accordance with $(4.5.5)$. Thus, the commutation rules $(4.5.11)$ and $(4.5.12)$ are supported.

4.6 The Hamiltonian of the quantized Maxwell field

We write the Hamiltonian $\boldsymbol{H}$ of the quantized electromagnetic field starting from $(4.2.9)$ and applying the quantization rule $(4.5.4\,/\,5)$ as follows

$$
\begin{aligned}
:\boldsymbol{H}: &= \int d^3x :\mathcal{H}(\underline{x}): \\
&= -\frac{1}{2\mu_0}\int d^3x : \sum_{\nu=0}^{3} g_{\nu\nu}\left[\left(\frac{d\boldsymbol{A}^{(\nu)}}{dx^0}\right)^2 + \sum_{m=1}^{3}\left(\frac{d\boldsymbol{A}^{(\nu)}}{dx^{(m)}}\right)^2\right]:.
\end{aligned}
\tag{4.6.1}
$$

We have taken the normal ordering for the integrand of $\boldsymbol{H}$ (set between colons). This ordering is explained after equation $(3.3.30)$. In contrast to the rules for Fermions here (for photons, which are Bosons) interchanging of the operators $\boldsymbol{a}_\lambda\left(\vec{k}\right)$ and $\boldsymbol{a}_{\lambda'}\left(\vec{k}'\right)$ does not need a change of the sign of the product. As shown in $(3.3.32)$ the normal ordering normalizes the energy operator $\boldsymbol{H}$ and eliminates the divergence as discussed in $(3.3.17)$ and $(3.3.25)$. The expansion $(4.5.9)$ implies

$$
\begin{aligned}
&\frac{d\boldsymbol{A}^{(\nu)}}{dx^0} \\
&= \frac{-\mathrm{i}}{\sqrt{\varepsilon_0\hbar}}\int \frac{d^3k}{\sqrt{2\omega_k(2\pi)^3}}\frac{\omega_k}{c\hbar}\sum_{\lambda=0}^{3}\varepsilon_{\nu,\lambda}\left(\vec{k}\right)\left(\boldsymbol{a}_\lambda\left(\vec{k}\right)e^{\mathrm{i}\left(\vec{k}\vec{x}-\omega_k t\right)/\hbar} - \boldsymbol{a}_\lambda^\dagger\left(\vec{k}\right)e^{-\mathrm{i}\left(\vec{k}\vec{x}-\omega_k t\right)/\hbar}\right),
\end{aligned}
\tag{4.6.2}
$$

$$
\begin{aligned}
&\frac{d\boldsymbol{A}^{(\nu)}}{dx^{(n)}} \\
&= \frac{\mathrm{i}}{\sqrt{\varepsilon_0\hbar}}\int \frac{d^3k}{\sqrt{2\omega_k(2\pi)^3}}\frac{k^{(n)}}{\hbar}\sum_{\lambda=0}^{3}\varepsilon_{\nu,\lambda}\left(\vec{k}\right)\left(\boldsymbol{a}_\lambda\left(\vec{k}\right)e^{\mathrm{i}\left(\vec{k}\vec{x}-\omega_k t\right)/\hbar} - \boldsymbol{a}_\lambda^\dagger\left(\vec{k}\right)e^{-\mathrm{i}\left(\vec{k}\vec{x}-\omega_k t\right)/\hbar}\right),
\end{aligned}
\tag{4.6.3}
$$

which we insert in $(4.6.1)$ like this

$$
\begin{aligned}
:\boldsymbol{H}: &= \frac{1}{\mu_0\varepsilon_0\hbar 2}\int d^3x \int \frac{d^3k'}{\sqrt{2\omega_{k'}(2\pi)^3}}\int \frac{d^3k}{\sqrt{2\omega_k(2\pi)^3}}\left(\frac{\omega_{k'}\omega_k}{c^2} + \vec{k}\cdot\vec{k}'\right) \\
&\cdot \frac{1}{\hbar^2}\sum_{\lambda',\lambda=0}^{3}\sum_{\nu=0}^{3} g_{\nu\nu}\varepsilon_{\nu,\lambda'}\left(\vec{k}'\right)\varepsilon_{\nu,\lambda}\left(\vec{k}\right) \\
&\cdot \Big(\boldsymbol{a}_{\lambda'}\left(\vec{k}'\right)\boldsymbol{a}_\lambda\left(\vec{k}\right)e^{\mathrm{i}\left(\left(\vec{k}'+\vec{k}\right)\vec{x}-(\omega_{k'}+\omega_k)t\right)/\hbar} + \boldsymbol{a}_{\lambda'}^\dagger\left(\vec{k}'\right)\boldsymbol{a}_\lambda^\dagger\left(\vec{k}\right)e^{-\mathrm{i}\left(\left(\vec{k}'+\vec{k}\right)\vec{x}-(\omega_{k'}+\omega_k)t\right)/\hbar} \\
&- \boldsymbol{a}_\lambda^\dagger\left(\vec{k}\right)\boldsymbol{a}_{\lambda'}\left(\vec{k}'\right)e^{\mathrm{i}\left(\left(\vec{k}'-\vec{k}\right)\vec{x}-(\omega_{k'}-\omega_k)t\right)/\hbar} - \boldsymbol{a}_{\lambda'}^\dagger\left(\vec{k}'\right)\boldsymbol{a}_\lambda\left(\vec{k}\right)e^{-\mathrm{i}\left(\left(\vec{k}'-\vec{k}\right)\vec{x}-(\omega_{k'}-\omega_k)t\right)/\hbar}\Big).
\end{aligned}
\tag{4.6.4}
$$

In the terms with $\int \frac{d^3k'}{\sqrt{2\omega_{k'}(2\pi)^3}}\left(\frac{\omega_{k'}\omega_k}{c^2}+\vec{k}'\cdot\vec{k}\right)\int d^3x\, e^{\pm i\left(\vec{k}'+\vec{k}\right)\vec{x}/\hbar}$ due to $(3.2.25)$ the vector $\vec{k}'$ equals to $-\vec{k}$ and therefore the value of the bracket above vanishes as a consequence of $(4.4.3)$. By the same reason in the term with $e^{\pm i\left(\vec{k}'-\vec{k}\right)\vec{x}/\hbar}$ the vector $\vec{k}'$ becomes $+\vec{k}$ and $\left(\frac{\omega_{k'}\omega_k}{c^2}+\vec{k}'\cdot\vec{k}\right)$ results in $\frac{2\omega_k^2}{c^2}$. Due to $(4.4.5)$ we have

$$\sum_{\nu=0}^{3} g_{\nu\nu}\varepsilon_{\nu,\lambda'}\left(\vec{k}\right)\varepsilon_{\nu,\lambda}\left(\vec{k}\right)=g_{\lambda'\lambda}\,.$$

Using $(4.1.9)$ we obtain

$$\begin{aligned}:\boldsymbol{H}: &= \int d^3k\;\omega_k\sum_{\lambda=0}^{3}\left(-g_{\lambda\lambda}\right)\boldsymbol{a}_\lambda^\dagger\left(\vec{k}\right)\boldsymbol{a}_\lambda\left(\vec{k}\right)\\ &= \int d^3k\;\omega_k\left(\sum_{\lambda=1}^{3}\left(\boldsymbol{a}_\lambda^\dagger\left(\vec{k}\right)\boldsymbol{a}_\lambda\left(\vec{k}\right)\right)-\boldsymbol{a}_0^\dagger\left(\vec{k}\right)\boldsymbol{a}_0\left(\vec{k}\right)\right).\end{aligned}\qquad(4.6.5)$$

In analogy to $(3.3.14)$ we interpret $\boldsymbol{a}_\lambda^\dagger\left(\vec{k}\right)\boldsymbol{a}_\lambda\left(\vec{k}\right)$ as a differential number operator i.e. $d^3k\,\boldsymbol{a}_\lambda^\dagger\left(\vec{k}\right)\boldsymbol{a}_\lambda\left(\vec{k}\right)$ generates the number of photon states in the area d^3k for the given λ as an eigenvalue. The term

$$-\boldsymbol{a}_0^\dagger\left(\vec{k}\right)\boldsymbol{a}_0\left(\vec{k}\right)\qquad(4.6.6)$$

in $(4.6.5)$ is problematic. Apparently, arbitrary amounts of energy could be gained by creating “scalar photons” with $\lambda=0$. We deal with this problem and denote a general state of photons by $\left|\Phi\right\rangle$ (state vector in Hilbert space.). In order or ascertain if its energy expectation value is independent of $\lambda=0$ and $\lambda=3$ there is a simple operator to which $\left|\Phi\right\rangle$ can be subjected (Gubta-Bleuler method). The operator reads

$$\begin{aligned}&\sum_{\mu=0}^{3}\frac{d\boldsymbol{A}_+^{(\mu)}\left(\underline{x}\right)}{dx^{(\mu)}}\quad\text{with}\\ &\boldsymbol{A}_+^{(\mu)}\left(\underline{x}\right)=\frac{1}{\sqrt{\varepsilon_0\hbar}}\int\frac{d^3k}{\sqrt{2\omega_k(2\pi)^3}}e^{i\left(\vec{k}\vec{x}-\omega_k t\right)/\hbar}\sum_{\lambda=0}^{3}\varepsilon_{\mu,\lambda}\left(\vec{k}\right)\boldsymbol{a}_\lambda\left(\vec{k}\right).\end{aligned}\qquad(4.6.7)$$

We claim that the constraint for $\left|\Phi\right\rangle$ to have the properties referred to is

$$\sum_{\mu=0}^{3}\frac{d\boldsymbol{A}_+^{(\mu)}}{dx}\left|\Phi\right\rangle=0\,.\qquad(4.6.8)$$

From $\omega_k=ck^0\equiv c\left|\vec{k}\right|$, $(4.4.3)$, and taking into account

$$t = \frac{x^0}{c}, \text{ i.e. } -\left(\vec{k}\vec{x} - \omega_k t\right) = \underline{kx}\ , \tag{4.6.9}$$

$(3.4.8)$ and $(3.4.35a)$, we can write

$$\sum_{\mu=0}^{3} \frac{d\boldsymbol{A}_{+}^{(\mu)}(\underline{x})}{dx^{(\mu)}} = \frac{1}{\sqrt{\varepsilon_0 \hbar}} \int \frac{d^3k}{\sqrt{2\omega_k (2\pi)^3}} e^{\mathrm{i}\underline{kx}/\hbar} \sum_{\lambda=0}^{3} \underline{k} \cdot \underline{\varepsilon}_{\lambda}\left(\vec{k}\right) \boldsymbol{a}_{\lambda}\left(\vec{k}\right). \tag{4.6.10}$$

Due to $\underline{k} = \left(\left|\vec{k}\right|, \vec{k}\right)$, $(4.4.6)$ and $(4.4.7)$, we have

$$\sum_{\lambda=0}^{3} \underline{k} \cdot \underline{\varepsilon}_{\lambda}\left(\vec{k}\right) \boldsymbol{a}_{\lambda}\left(\vec{k}\right) = \left|\vec{k}\right| \boldsymbol{a}_0\left(\vec{k}\right) + 0 + 0 - \frac{\vec{k} \cdot \vec{k}}{\left|\vec{k}\right|} \boldsymbol{a}_3\left(\vec{k}\right)$$

and therefore according to $(4.6.8)$

$$\sum_{\mu=0}^{3} \frac{d\boldsymbol{A}_{+}^{(\mu)}(\underline{x})}{dx^{(\mu)}} \left|\Phi\right\rangle = \frac{1}{\sqrt{\varepsilon_0 \hbar}} \int \frac{d^3k}{\sqrt{2\omega_k (2\pi)^3}} e^{-\mathrm{i}\underline{kx}/\hbar} \left|\vec{k}\right| \left(\boldsymbol{a}_0\left(\vec{k}\right) - \boldsymbol{a}_3\left(\vec{k}\right)\right) \left|\Phi\right\rangle = 0\,. \tag{4.6.11}$$

This equation implies

$$\left(\boldsymbol{a}_0\left(\vec{k}\right) - \boldsymbol{a}_3\left(\vec{k}\right)\right)\left|\Phi\right\rangle = 0, \text{ i.e. } \boldsymbol{a}_0\left(\vec{k}\right)\left|\Phi\right\rangle = \boldsymbol{a}_3\left(\vec{k}\right)\left|\Phi\right\rangle \tag{4.6.12}$$

with the hermitean adjoint relation

$$\left\langle\Phi\right|\left(\boldsymbol{a}_0^{\dagger}\left(\vec{k}\right) - \boldsymbol{a}_3^{\dagger}\left(\vec{k}\right)\right) = 0, \text{ i.e. } \left\langle\Phi\right|\boldsymbol{a}_0^{\dagger}\left(\vec{k}\right) = \left\langle\Phi\right|\boldsymbol{a}_3^{\dagger}\left(\vec{k}\right). \tag{4.6.13}$$

We form the expectation values

$$\begin{aligned} &\left\langle\Phi\right|\boldsymbol{a}_0^{\dagger}\left(\vec{k}\right)\boldsymbol{a}_0\left(\vec{k}\right)\left|\Phi\right\rangle = \left\langle\Phi\right|\boldsymbol{a}_3^{\dagger}\left(\vec{k}\right)\boldsymbol{a}_3\left(\vec{k}\right)\left|\Phi\right\rangle, \\ &\text{i.e. } \left\langle\Phi\right|\left(\boldsymbol{a}_0^{\dagger}\left(\vec{k}\right)\boldsymbol{a}_0\left(\vec{k}\right) - \boldsymbol{a}_3^{\dagger}\left(\vec{k}\right)\boldsymbol{a}_3\left(\vec{k}\right)\right)\left|\Phi\right\rangle = 0 \end{aligned} \tag{4.6.14}$$

and using $(4.6.5)$ we make up

$$\begin{aligned} \left\langle\Phi\right|\boldsymbol{H}\left|\Phi\right\rangle &= \int d^3k\,\omega_k \Big(\sum_{\lambda=1}^{2}\left\langle\Phi\right|\boldsymbol{a}_{\lambda}^{\dagger}\left(\vec{k}\right)\boldsymbol{a}_{\lambda}\left(\vec{k}\right)\left|\Phi\right\rangle \\ &\quad + \left\langle\Phi\right|\left(\boldsymbol{a}_3^{\dagger}\left(\vec{k}\right)\boldsymbol{a}_3\left(\vec{k}\right) - \boldsymbol{a}_0^{\dagger}\left(\vec{k}\right)\boldsymbol{a}_0\left(\vec{k}\right)\right)\left|\Phi\right\rangle\Big) \\ &= \int d^3k\,\omega_k \sum_{\lambda=1}^{2}\left\langle\Phi\right|\boldsymbol{a}_{\lambda}^{\dagger}\left(\vec{k}\right)\boldsymbol{a}_{\lambda}\left(\vec{k}\right)\left|\Phi\right\rangle \end{aligned} \tag{4.6.15}$$

were $(4.6.14)$ has been inserted.

We denote the eigenvalue of the differential number operator $\boldsymbol{a}_\lambda^\dagger(\vec{k})\boldsymbol{a}_\lambda(\vec{k})$ on $|\Phi\rangle$ by $n_\lambda(\vec{k})$ and obtain

$$\langle\Phi|\boldsymbol{H}|\Phi\rangle = \int d^3k\ \omega_k \sum_{\lambda=1}^{2} n_\lambda(\vec{k}). \qquad (4.6.16)$$

As demanded only $\lambda = 1$ and $\lambda = 2$ appear if the condition $(4.6.8)$ is fulfilled.

4.7 The Feynman propagator for photons

As mentioned in section 3.4 the Feynman propagator will be used for the formulation of interactions. In analogy to $(3.4.4)$ it is defined as follows

$$\mathrm{i}D_F^{\mu\nu}(\underline{x}-\underline{y}) = \langle 0|T\left(\boldsymbol{A}^{(\mu)}(\underline{x})\boldsymbol{A}^{(\nu)}(\underline{y})\right)|0\rangle \qquad (4.7.1)$$

with the following time-ordered product (as $(3.4.2)$)

$$T\left(\boldsymbol{A}^{(\mu)}(x^0,\vec{x})\boldsymbol{A}^{(\nu)}(y^0,\vec{y})\right) = \begin{Bmatrix} \boldsymbol{A}^{(\mu)}(x^0,\vec{x})\boldsymbol{A}^{(\nu)}(y^0,\vec{y}) & \text{for } x^0 > y^0 \\ \boldsymbol{A}^{(\nu)}(y^0,\vec{y})\boldsymbol{A}^{(\mu)}(x^0,\vec{x}) & \text{for } y^0 > x^0 \end{Bmatrix}. \qquad (4.7.2)$$

In contrast to Fermions, c.f. $(3.4.2)$, by definition time ordering of bosonic operators does not change the sign. In the context to $(4.6.16)$ or to $(2.2.20)$ the expression $\boldsymbol{a}_\lambda^\dagger(\vec{k})\boldsymbol{a}_\lambda(\vec{k})$ is interpreted as a number operator. Consequently

$$\boldsymbol{a}_\lambda^\dagger(\vec{k})\boldsymbol{a}_\lambda(\vec{k})|0\rangle = 0 \ \text{ and therefore } \boldsymbol{a}_\lambda(\vec{k})|0\rangle = 0 \qquad (4.7.3)$$

hold in analogy to $(3.3.20)$. The hermitean adjoint reads as in $(3.4.7)$

$$\langle 0|\boldsymbol{a}_\lambda^\dagger(\vec{k}) = 0. \qquad (4.7.4)$$

Temporarily we set $x^0 > y^0$. Making use of $(4.5.9)$, $(4.6.9)$, $(4.7.3)$ and $(4.7.4)$ we calculate

$$\begin{aligned}\langle 0|\boldsymbol{A}^{(\mu)}(\underline{x})\boldsymbol{A}^{(\nu)}(\underline{y})|0\rangle &= \langle 0|\frac{1}{\varepsilon_0\hbar}\int\frac{d^3k}{\sqrt{2\omega_k(2\pi)^3}}\int\frac{d^3k'}{\sqrt{2\omega_{k'}(2\pi)^3}} \\ &\cdot\sum_{\lambda,\lambda'=0}^{3}\boldsymbol{a}_\lambda(\vec{k})\,\boldsymbol{a}_{\lambda'}^\dagger(\vec{k}')\varepsilon_{\mu,\lambda}(\vec{k})\varepsilon_{\nu,\lambda'}(\vec{k}')e^{\mathrm{i}(-\underline{k}\underline{x}+\underline{k}'\underline{y})/\hbar}|0\rangle.\end{aligned} \qquad (4.7.5)$$

We insert $(4.5.11)$ using

$$\langle 0|\boldsymbol{a}_\lambda(\vec{k})\boldsymbol{a}_{\lambda'}^\dagger(\vec{k}')|0\rangle = \langle 0|\left[\boldsymbol{a}_\lambda(\vec{k})\boldsymbol{a}_{\lambda'}^\dagger(\vec{k}')\right]_-|0\rangle - g_{\lambda\kappa'}\delta^3(\vec{k}-\vec{k}')$$

like this

$$\langle 0|\boldsymbol{A}^{(\mu)}(\underline{x})\boldsymbol{A}^{(\nu)}(\underline{y})|0\rangle$$

$$=\frac{1}{\varepsilon_0\hbar}\int\frac{d^3k}{2\omega_k(2\pi)^3}\sum_{\lambda=0}^{3}(-g_{\lambda\lambda})\,\varepsilon_{\mu,\lambda}(\vec{k})\varepsilon_{\nu,\lambda}(\vec{k})e^{-\mathrm{i}\underline{k}(\underline{x}-\underline{y})/\hbar} \tag{4.7.6}$$

With the help of $(4.5.14)$ the expression $(4.7.1)$ becomes

$$\begin{aligned}\mathrm{i}D_F^{\mu\nu}(\underline{x}-\underline{y}) &= \frac{1}{\varepsilon_0\hbar}\int\frac{d^3k}{2\omega_k(2\pi)^3}g_{\mu\nu}\begin{Bmatrix} e^{-\mathrm{i}\underline{k}(\underline{x}-\underline{y})/\hbar} & \text{for } x^0>y^0 \\ e^{\mathrm{i}\underline{k}(\underline{x}-\underline{y})/\hbar} & \text{for } \mathrm{y}^0>x^0\end{Bmatrix}\\ &= \frac{-g_{\mu\nu}}{\varepsilon_0\hbar}\int\frac{d^3q\hbar^3}{2q^0c\hbar(2\pi^3)}e^{\mp\mathrm{i}\underline{q}(\underline{x}-\underline{y})}\\ &= \frac{-g_{\mu\nu}\hbar}{\varepsilon_0 c}\int\frac{d^3q}{2q^0(2\pi^3)}e^{\mp\mathrm{i}\underline{q}(\underline{x}-\underline{y})},\end{aligned} \tag{4.7.7}$$

where we have used $\underline{q}=\underline{k}/\hbar$ and $q^0=\omega_k/(c\hbar)$. The expression $(4.7.7)$ contains the scalar Feynman propagator $\Delta_F(\underline{x}-\underline{y})$, $(3.4.35)$. According to $(3.4.22)$ we can write

$$D_F^{\mu\nu}(\underline{x}-\underline{y})=(-g_{\mu\nu})\Delta_F(\underline{x}-\underline{y})\frac{\hbar}{\varepsilon_0 c} \tag{4.7.8}$$

Due to $(3.4.36)$ setting the mass $m=0$ the Feynman propagator for photons finally reads

$$D_F^{\mu\nu}(\underline{x}-\underline{y})=-g_{\mu\nu}\int\frac{d^4q}{(2\pi)^4}\frac{e^{-\mathrm{i}\underline{q}(\underline{x}-\underline{y})}}{\underline{q}^2+\mathrm{i}\varepsilon}\frac{\hbar}{\varepsilon_0 c}\,. \tag{4.7.9}$$

According to $(3.4.37)$ the Fourier transformed propagator is

$$D_F^{\mu\nu}(\underline{q})=\frac{-g_{\mu\nu}}{\underline{q}^2+\mathrm{i}\varepsilon}\frac{\hbar}{\varepsilon_0 c}\,. \tag{4.7.10}$$

5 Interacting quantum fields

5.1 The interaction picture

As a starting point for the development of a perturbation theory one assumes that the Hamiltonian of a system can be split into two parts

$$\boldsymbol{H} = \boldsymbol{H}_0 + \boldsymbol{H}_1 . \tag{5.1.1}$$

Here $\boldsymbol{H}_0$ is the Hamiltonian of the system without interactions, i.e. of a system of free fields. For this Hamiltonian an exact solution can be given. If the “perturbation operator” $\boldsymbol{H}_1$ represents an interaction with a small strength, it is possible to obtain reliable approximate solutions.

The foundation of our quantum mechanical developments is Schrödinger’s theory. If $|\alpha,t\rangle$ is a state (state vector) in the so-called “Schrödinger picture” the Schrödinger equation holds

$$\mathrm{i}\hbar\frac{d}{dt}|\alpha,t\rangle \equiv \boldsymbol{H}|\alpha,t\rangle = E|\alpha,t\rangle . \tag{5.1.2}$$

We will show that a similar relation holds for $\boldsymbol{H}_1$. To this end we transform every operator $\boldsymbol{O}$ by a unitary transformation like this

$$\boldsymbol{O}^I(t) = e^{\mathrm{i}\boldsymbol{H}_0 t/\hbar}\boldsymbol{O}e^{-\mathrm{i}\boldsymbol{H}_0 t/\hbar} \tag{5.1.3}$$

with the definition $$e^{\mathrm{i}\boldsymbol{H}_0 t/\hbar} = \boldsymbol{1} + \mathrm{i}\boldsymbol{H}_0 t/\hbar + \frac{1}{2!}\left(\mathrm{i}\boldsymbol{H}_0 t/\hbar\right)^2 + \cdots . \tag{5.1.4}$$

At the same time we transform the state $|\alpha,t\rangle$

$$|\alpha,t\rangle^I = e^{\mathrm{i}\boldsymbol{H}_0 t/\hbar}|\alpha,t\rangle . \tag{5.1.5}$$

By $(5.1.3)$ and $(5.1.5)$ the quantities are transformed in a system named “interaction picture”. The hermitean adjoint of $(5.1.5)$ reads

$$^I\langle\alpha,t| = \langle\alpha,t|e^{-\mathrm{i}\boldsymbol{H}_0 t/\hbar} \tag{5.1.6}$$

because $\boldsymbol{H}_0$ is a hermitean operator. We calculate the following matrix element

$$\begin{aligned} ^I\langle\beta,t|\boldsymbol{O}^I|\alpha,t\rangle^I &= \langle\beta,t|e^{-\mathrm{i}\boldsymbol{H}_0 t/\hbar}e^{\mathrm{i}\boldsymbol{H}_0 t/\hbar}\boldsymbol{O}e^{-\mathrm{i}\boldsymbol{H}_0 t/\hbar}e^{\mathrm{i}\boldsymbol{H}_0 t/\hbar}|\alpha,t\rangle \\ &= \langle\beta,t|\boldsymbol{O}|\alpha,t\rangle . \end{aligned} \tag{5.1.7}$$

Thus, the matrix element is identical in both “pictures”. Because physical observables are matrix elements both “pictures” are equivalent. We now deal with the following derivative using $(5.1.1)$ up to $(5.1.5)$

$$\mathrm{i}\hbar\frac{d}{dt}\left|\alpha,t\right\rangle^{I} = \mathrm{i}\hbar\frac{d}{dt}\left(e^{\mathrm{i}\boldsymbol{H}_0 t/\hbar}\left|\alpha,t\right\rangle\right) = -\boldsymbol{H}_0 e^{\mathrm{i}\boldsymbol{H}_0 t/\hbar}\left|\alpha,t\right\rangle + e^{\mathrm{i}\boldsymbol{H}_0 t/\hbar}\mathrm{i}\hbar\frac{d}{dt}\left|\alpha,t\right\rangle$$
$$= -\boldsymbol{H}_0 e^{\mathrm{i}\boldsymbol{H}_0 t/\hbar}\left|\alpha,t\right\rangle + e^{\mathrm{i}\boldsymbol{H}_0 t/\hbar}\boldsymbol{H}\left|\alpha,t\right\rangle = e^{\mathrm{i}\boldsymbol{H}_0 t/\hbar}\left(-\boldsymbol{H}_0 + \boldsymbol{H}_0 + \boldsymbol{H}_1\right)\left|\alpha,t\right\rangle \qquad (5.1.8)$$
$$= e^{\mathrm{i}\boldsymbol{H}_0 t/\hbar}\boldsymbol{H}_1 e^{-\mathrm{i}\boldsymbol{H}_0 t/\hbar}e^{\mathrm{i}\boldsymbol{H}_0 t/\hbar}\left|\alpha,t\right\rangle = \boldsymbol{H}_1^I\left|\alpha,t\right\rangle^I .$$

As claimed in context with $(5.1.2)$, equation $(5.1.8)$ shows that in the "interaction picture" a relation holds for the "perturbation Hamiltonian" $\boldsymbol{H}_1^I$ which is similar to the Schrödinger equation $(5.1.2)$

5.2 The time evolution operator

We stay in the "interaction picture" and investigate now how a state $\left|\alpha,t\right\rangle^I$ is coming along in the course of the time. We define the time evolution operator $\boldsymbol{U}(t_1,t_0)$ which describes the connection between the state vectors at the times t_0 and t_1 i.e.

$$\left|\alpha,t_1\right\rangle^I = \boldsymbol{U}(t_1,t_0)\left|\alpha,t_0\right\rangle^I . \qquad (5.2.1)$$

On the other hand $(5.1.5)$ reads $\left|\alpha,t_1\right\rangle^I = e^{\mathrm{i}\boldsymbol{H}_0 t_1/\hbar}\left|\alpha,t_1\right\rangle$. (5.2.2)

The Schrödinger equation $(5.1.2)$ says

$$\mathrm{i}\hbar\frac{d}{dt_1}\left|\alpha,t_1\right\rangle = \boldsymbol{H}\left|\alpha,t_1\right\rangle \qquad (5.2.3)$$

and has the solution $\left|\alpha,t_1\right\rangle = e^{-\mathrm{i}\boldsymbol{H}(t_1-t_0)/\hbar}\left|\alpha,t_0\right\rangle$. (5.2.4)

We insert $(5.2.4)$ in $(5.2.2)$ like this

$$\left|\alpha,t_1\right\rangle^I = e^{\mathrm{i}\boldsymbol{H}_0 t_1/\hbar}e^{-\mathrm{i}\boldsymbol{H}(t_1-t_0)/\hbar}\left|\alpha,t_0\right\rangle$$
$$= e^{\mathrm{i}\boldsymbol{H}_0 t_1/\hbar}e^{-\mathrm{i}\boldsymbol{H}(t_1-t_0)/\hbar}e^{-\mathrm{i}\boldsymbol{H}_0 t_0/\hbar}e^{\mathrm{i}\boldsymbol{H}_0 t_0/\hbar}\left|\alpha,t_0\right\rangle \qquad (5.2.5)$$
$$= e^{\mathrm{i}\boldsymbol{H}_0 t_1/\hbar}e^{-\mathrm{i}\boldsymbol{H}(t_1-t_0)/\hbar}e^{-\mathrm{i}\boldsymbol{H}_0 t_0/\hbar}\left|\alpha,t_0\right\rangle^I ,$$

where $(5.1.5)$ has been used. Comparing $(5.2.1)$ with $(5.2.5)$ yields

$$\boldsymbol{U}(t_1,t_0) = e^{\mathrm{i}\boldsymbol{H}_0 t_1/\hbar}e^{-\mathrm{i}\boldsymbol{H}(t_1-t_0)/\hbar}e^{-\mathrm{i}\boldsymbol{H}_0 t_0/\hbar}. \qquad (5.2.6)$$

Obviously $\boldsymbol{U}(t_0,t_0) = \boldsymbol{1}$ holds. (5.2.7)

Now we start with $(5.1.8)$

$$\mathrm{i}\hbar\frac{d}{dt}\left|\alpha,t\right\rangle^I = \boldsymbol{H}_1^I\left|\alpha,t\right\rangle^I$$

and insert $(5.2.1)$ as follows

$$\begin{gathered} \mathrm{i}\hbar \frac{d}{dt} \boldsymbol{U}(t,t_0) |\alpha,t_0\rangle^I = \boldsymbol{H}_1^I \boldsymbol{U}(t,t_0) |\alpha,t_0\rangle^I \quad \text{or} \\ \mathrm{i}\hbar \frac{d}{dt} \boldsymbol{U}(t,t_0) = \boldsymbol{H}_1^I \boldsymbol{U}(t,t_0). \end{gathered} \tag{5.2.8}$$

We transform this differential equation into an integral equation which meets $(5.2.8)$

$$\boldsymbol{U}(t,t_0) = \boldsymbol{1} + \frac{(-\mathrm{i})}{\hbar} \int_{t_0}^{t} dt' \boldsymbol{H}_1^I(t') \boldsymbol{U}(t',t_0), \tag{5.2.9}$$

which is verified by differentiating with respect to t. We insert $(5.2.9)$ repeatedly in itself like this

$$\begin{aligned} &\boldsymbol{U}(t,t_0) = \boldsymbol{1} + (-\mathrm{i}) \\ &\cdot \int_{t_0}^{t} dt_1 \frac{\boldsymbol{H}_1^I(t_1)}{\hbar} \left(\boldsymbol{1} + (-\mathrm{i}) \int_{t_0}^{t_1} dt_2 \frac{\boldsymbol{H}_1^I(t_2)}{\hbar} \left(\boldsymbol{1} + (-\mathrm{i}) \int_{t_0}^{t_2} dt_3 \frac{\boldsymbol{H}_1^I(t_3)}{\hbar} (\cdots) \right) \cdots \right) \\ &= \boldsymbol{1} + (-\mathrm{i}) \int_{t_0}^{t} dt_1 \frac{\boldsymbol{H}_1^I(t_1)}{\hbar} \\ &+ (-\mathrm{i})^2 \int_{t_0}^{t} dt_1 \frac{\boldsymbol{H}_1^I(t_1)}{\hbar} \int_{t_0}^{t_1} dt_2 \frac{\boldsymbol{H}_1^I(t_2)}{\hbar} \\ &\vdots \\ &+ (-\mathrm{i})^n \int_{t_0}^{t} dt_1 \frac{\boldsymbol{H}_1^I(t_1)}{\hbar} \int_{t_0}^{t_1} dt_2 \frac{\boldsymbol{H}_1^I(t_2)}{\hbar} \int_{t_0}^{t_2} dt_3 \frac{\boldsymbol{H}_1^I(t_3)}{\hbar} \cdots \int_{t_0}^{t_{n-1}} dt_n \frac{\boldsymbol{H}_1^I(t_n)}{\hbar} \\ &\vdots \end{aligned} \tag{5.2.10}$$

with $n \to \infty$. From now on the index I will be dropped since all the results in this chapter will be based on the interaction picture.

Apparently in $(5.2.10)$ is $t_k \geq t_{k+1}$. However, in the terms of second and higher order the products of the $\boldsymbol{H}_1$'s can be written in arbitrary order provided that a time ordering operator T acts on them. This operator was introduced in $(3.4.2)$ and $(4.7.2)$. The upper limit of every following integral has to agree with the argument of the preceding integrand. Time ordering means here

$$\begin{aligned} &T\left(\boldsymbol{H}_1(t_1)\boldsymbol{H}_1(t_2)\cdots\boldsymbol{H}_1(t_n)\right) = \boldsymbol{H}_1(t_1')\boldsymbol{H}_1(t_2')\cdots\boldsymbol{H}_1(t_n'), \\ &\text{where } t_1' \geq t_2' \geq \cdots \geq t_n' \text{ and } (t_1',t_2',\cdots,t_n') \text{ is a} \\ &\text{permutation of } (t_1,t_2,\cdots,t_n). \end{aligned} \tag{5.2.11}$$

In $(5.2.10)$ the upper boundaries are mutually dependent, which makes these multiple integrals quite difficult to handle. Fortunately, however, following an idea by Dyson (Dyson, 1949) the integration can be written such, that they all cover the full time interval $[t_0, t]$. For this let us investigate the term of second order in the series $(5.2.10)$. Because of $t_1 \geq t_2$ we can write

$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \boldsymbol{H}_1(t_1)\boldsymbol{H}_1(t_2) = \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 T\left(\boldsymbol{H}_1(t_1)\boldsymbol{H}_1(t_2)\right). \qquad (5.2.12)$$

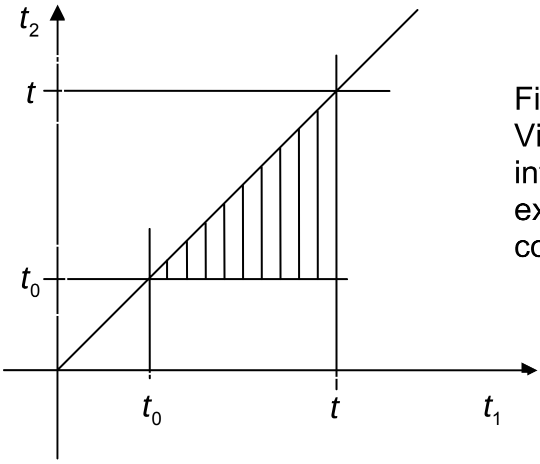

Figure 5.2.1
Visualization of the integrations in the expression (5.2.12). They cover a triangular area.

In figure 5.2.1 the integration $(5.2.12)$ is shown, which extends over a triangular area in the t_1-t_2-plane.

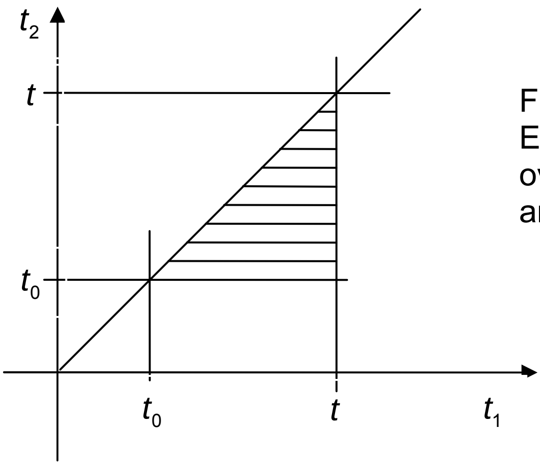

Figure 5.2.2
Equivalent way to integrate over the same triangular area as in figure 5.2.1.

As sketched in figure 5.2.2, when the boundaries are suitably chosen one may as well integrate first over t_1 and then over t_2, i.e.

$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \, \boldsymbol{H}_1(t_1)\boldsymbol{H}_1(t_2) = \int_{t_0}^{t} dt_2 \int_{t_1}^{t} dt_1 \boldsymbol{H}_1(t_1)\boldsymbol{H}_1(t_2), \qquad (5.2.13)$$

where $t_1 \geq t_2$ holds. On the right hand side we rename the integration variables $t_1 \rightleftarrows t_2$ i.e.

$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \, \boldsymbol{H}_1(t_1)\, \boldsymbol{H}_1(t_2) = \int_{t_0}^{t} dt_1 \int_{t_1}^{t} dt_2 \, \boldsymbol{H}_1(t_2)\, \boldsymbol{H}_1(t_1) \tag{5.2.14}$$

with $t_2 \geq t_1$ on the right hand side. Therefore, $(5.2.14)$ is equivalent to

$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \, \boldsymbol{H}_1(t_1)\, \boldsymbol{H}_1(t_2) = \int_{t_0}^{t} dt_1 \int_{t_1}^{t} dt_2 \, T\left(\boldsymbol{H}_1(t_1)\boldsymbol{H}_1(t_2)\right). \tag{5.2.15}$$

We add the equations $(5.2.12)$ and $(5.2.15)$ as follows

$$\begin{aligned} 2\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \, \boldsymbol{H}_1(t_1)\boldsymbol{H}_1(t_2) &= \int_{t_0}^{t} dt_1 \left(\int_{t_0}^{t_1} dt_2 + \int_{t_1}^{t} dt_2 \right) T\left(\boldsymbol{H}_1(t_1)\boldsymbol{H}_1(t_2)\right) \\ &= \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 T\left(\boldsymbol{H}_1(t_1)\boldsymbol{H}_1(t_2)\right). \end{aligned} \tag{5.2.16}$$

We have achieved that the boundaries of both integrals are identical. It can be shown that $(5.2.16)$ gets generalized into

$$n!\int_{t_0}^{t} dt_1 \cdots \int_{t_0}^{t_{n-1}} dt_n \boldsymbol{H}_1(t_1)\cdots \boldsymbol{H}_1(t_n) = \int_{t_0}^{t} dt_1 \cdots \int_{t_0}^{t} dt_n T\left(\boldsymbol{H}_1(t_1)\cdots \boldsymbol{H}_1(t_n)\right). \tag{5.2.17}$$

Inserting $(5.2.17)$ in $(5.2.10)$ results in

$$\boldsymbol{U}(t,t_0) = \boldsymbol{1} + \sum_{n=1}^{\infty} \frac{1}{n!}(-\mathrm{i})^n \int_{t_0}^{t} dt_1 \cdots \int_{t_0}^{t} dt_n T\left(\frac{\boldsymbol{H}_1(t_1)}{\hbar} \cdots \frac{\boldsymbol{H}_1(t_n)}{\hbar} \right). \tag{5.2.18}$$

We verify that the expression $(5.2.18)$ solves the original differential equation $(5.2.8)$. Taking the derivation with respect to t we get using $(5.2.11)$

$$\begin{aligned} \mathrm{i}\hbar \frac{d}{dt}\boldsymbol{U}(t,t_0) &= \mathrm{i}\hbar \sum_{n=1}^{\infty} \frac{1}{n!}(-\mathrm{i})^n \frac{d}{dt}\int_{t_0}^{t} dt_1 \cdots \int_{t_0}^{t} dt_n T\left(\frac{\boldsymbol{H}_1(t_1)}{\hbar} \cdots \frac{\boldsymbol{H}_1(t_n)}{\hbar} \right) \\ &= \mathrm{i}\sum_{n=1}^{\infty} \frac{1}{n!}(-\mathrm{i})^n \, n\boldsymbol{H}_1(t) \int_{t_0}^{t} dt_1 \cdots \int_{t_0}^{t} dt_{n-1} T\left(\frac{\boldsymbol{H}_1(t_1)}{\hbar} \cdots \frac{\boldsymbol{H}_1(t_{n-1})}{\hbar} \right) \\ &= \sum_{n=1}^{\infty} \frac{1}{(n-1)!}(-\mathrm{i})^{n-1} \boldsymbol{H}_1(t) \int_{t_0}^{t} dt_1 \cdots \int_{t_0}^{t} dt_{n-1} \, T\left(\frac{\boldsymbol{H}_1(t_1)}{\hbar} \cdots \frac{\boldsymbol{H}_1(t_{n-1})}{\hbar} \right) \\ &= \sum_{n'=0}^{\infty} \frac{1}{n'!}(-\mathrm{i})^{n'} \boldsymbol{H}_1(t) \int_{t_0}^{t} dt_1 \cdots \int_{t_0}^{t} dt_{n'} T\left(\frac{\boldsymbol{H}_1(t_1)}{\hbar} \cdots \frac{\boldsymbol{H}_1(t_{n'})}{\hbar} \right) \\ &= \boldsymbol{H}_1(t)\boldsymbol{U}(t,t_0). \end{aligned} \tag{5.2.19}$$

Here, in the first step we obtain n times the factor

$$\frac{d}{dt}\int_{t_0}^{t} dt'_k \boldsymbol{H}_1(t'_k) = \boldsymbol{H}_1(t), \quad \text{k=1,}\cdots\text{,n,}$$

and then $\mathrm{i}(-\mathrm{i})^n = \mathrm{i}(-\mathrm{i})(-\mathrm{i})^{n-1} = (-\mathrm{i})^{n-1}$. In the last step the term with $n' = 0$ equals to $\boldsymbol{1} \cdot \boldsymbol{H}_1(t) / \hbar$.

The main use of the time-evolution operator lies in its application to scattering processes.

5.3 The scattering matrix

The scattering matrix (S matrix) is a central concept in quantum field theory as well as in ordinary quantum mechanics. It describes the probability amplitude for a process in which the system makes a transition from an initial to a final state under the influence of an interaction. If one works in the "interaction picture" the time-evolution operator is the right tool to use to evaluate the scattering matrix. In constructing the S matrix we will proceed in a naïve way. Some problems with this approach are discussed in Greiner, Reinhardt, 1996, p. 269.

The scattering processes start with an experimentally given flux of particles or quantas with target particles. This state is named $|\Phi_i\rangle$, where the index i comprises its quantum numbers, and is given at the time $t \to -\infty$ with respect to the interaction process. The state of the system at the time t subject to the interaction is called $|\Psi(t)\rangle$. Therefore

$$\lim_{t\to-\infty} |\Psi(t)\rangle = |\Phi_i\rangle \tag{5.3.1}$$

holds. In the scattering experiments a specific final state $|\Phi_f\rangle$ with the quantum numbers f is selected. We now want to formulate how strongly the asymptotic state

$$\lim_{t\to+\infty} |\Psi(t)\rangle \tag{5.3.2}$$

agrees with the mentioned final state $|\Phi_f\rangle$, i.e. we are interested in the amplitude with which $|\Phi_f\rangle$ is present in $\lim_{t\to+\infty} |\Psi(t)\rangle$. In other words, we are looking for the projection S_{fi} of the state $(5.3.2)$ on the specific final state $|\Phi_f\rangle$, i.e.

$$S_{fi} = \lim_{t\to\infty} \langle\Phi_f|\Psi(t)\rangle. \tag{5.3.3}$$

We define $\boldsymbol{S}$ as the operator which transforms the initial state $|\Phi_i\rangle$ to the asymptotic state $(5.3.2)$ influenced by the interaction, i.e.

$$\lim_{t\to\infty} |\Psi(t)\rangle = \boldsymbol{S}|\Phi_i\rangle. \tag{5.3.4}$$

Thus, equation $(5.3.3)$ becomes

$$S_{fi} = \langle \Phi_f | \boldsymbol{S} | \Phi_i \rangle . \tag{5.3.5}$$

The operator $\boldsymbol{S}$ transforms from the time $t = -\infty$ to the time $t = +\infty$. Therefore, by means of $(5.2.1)$ we state

$$\boldsymbol{S} = \boldsymbol{U}(\infty, -\infty) . \tag{5.3.6}$$

We insert $(5.2.18)$ like this

$$\boldsymbol{S} = \sum_{n=0}^{\infty} \frac{1}{n!} (-\mathrm{i})^n \int_{-\infty}^{\infty} dt_1 \cdots \int_{-\infty}^{\infty} dt_n T \left(\frac{\boldsymbol{H}_1(t_1)}{\hbar} \cdots \frac{\boldsymbol{H}_1(t_n)}{\hbar} \right) \tag{5.3.7}$$

where again the term with $n = 0$ equals to $\boldsymbol{1}$.

Because the Hamiltonians of free Dirac particles, $(3.3.25)$, and of free photons, $(4.6.5)$, are linear combinations of products of a creating and an annihilating operator, the interaction Hamiltonians $\boldsymbol{H}_1(t)$ are more complicated superpositions of these operators and the terms in $\boldsymbol{S}$ become more complex with growing order n in $(5.3.7)$. In the next section we will develop tools which allow us to approach this task in an economical way.

If we write the Hamiltonian $\boldsymbol{H}_1(t_k)$ with the help of the Hamiltonian density $\mathcal{H}_1(t_k)$ as follows

$$\boldsymbol{H}_1(t_k) = \int d^3x \mathcal{H}_1(t_k)$$

the expression $(5.3.7)$ gets into

$$\boldsymbol{S} = \sum_{n=0}^{\infty} \frac{1}{n!} (-\mathrm{i})^n \int_{-\infty}^{\infty} \frac{d^4x_1}{c} \cdots \frac{d^4x_n}{c} T \left(\frac{\mathcal{H}_1(t_1)}{\hbar} \cdots \frac{\mathcal{H}_1(t_n)}{\hbar} \right) . \tag{5.3.8}$$

Knowing the S matrix elements of observable quantities, we can, for example, calculate scattering cross sections and decay rates by taking the square of these elements and performing some kinematical manipulations.

5.4 Wick's theorem

As mentioned in the last section, the calculation of the $\boldsymbol{S}$ operator or of the S matrix element S_{fi}, $(5.3.5)$, can be very laborious. Especially the "time ordering" in $(5.3.8)$ makes it rather confusing. Fortunately, there is a very elegant and efficient tool that can be used to systematically evaluate any complicated time-ordered product. This is Wick's theorem, which we will derive in the following.

We will show that "time ordering" is closely connected with the "normal product", which was introduced in section 3.3 and used in $(4.6.1)$. First, we split the field operators, which we name generally $\boldsymbol{\varphi}(\underline{x})$ in two parts like this

$$\boldsymbol{\varphi}(\underline{x}) = \boldsymbol{\varphi}^{+}(\underline{x}) + \boldsymbol{\varphi}^{-}(\underline{x}) . \qquad (5.4.1)$$

Specially for Dirac particles, $(3.3.29)$, which we transform to box normalization using $(3.3.30)$, we have the following partial field operators

$$\begin{aligned}
\underline{\boldsymbol{\varphi}}^{+}(\underline{x}) &= \sum_{r=1}^{2}\int \frac{d^3p}{\mathcal{E}\sqrt{V}}\sqrt{\frac{mc^2}{E(\vec{p})}}\boldsymbol{b}_r(\vec{p})\underline{w}_r(\vec{p})e^{\mathrm{i}\left(\vec{p}\vec{x}-E(\vec{p})x^0/c\right)/\hbar},\\
\underline{\boldsymbol{\varphi}}^{-}(\underline{x}) &= \sum_{r=1}^{2}\int \frac{d^3p}{\mathcal{E}\sqrt{V}}\sqrt{\frac{mc^2}{E(\vec{p})_r}}\boldsymbol{d}_r^{\dagger}(\vec{p})\underline{v}_r(\vec{p})e^{-\mathrm{i}\left(\vec{p}\vec{x}-E(\vec{p})x^0/c\right)/\hbar},\\
\underline{\boldsymbol{\varphi}}^{\dagger +}(\underline{x}) &= \sum_{r=1}^{2}\int \frac{d^3p}{\mathcal{E}\sqrt{V}}\sqrt{\frac{mc^2}{E(\vec{p})_r}}\boldsymbol{d}_r(\vec{p})\underline{v}_r^{\dagger}(\vec{p})e^{\mathrm{i}\left(\vec{p}\vec{x}-E(\vec{p})x^0/c\right)/\hbar}\\
\underline{\boldsymbol{\varphi}}^{\dagger -}(\underline{x}) &= \sum_{r=1}^{2}\int \frac{d^3p}{\mathcal{E}\sqrt{V}}\sqrt{\frac{mc^2}{E(\vec{p})}}\boldsymbol{b}_r^{\dagger}(\vec{p})\underline{w}_r^{\dagger}(\vec{p})e^{-\mathrm{i}\left(\vec{p}\vec{x}-E(\vec{p})x^0/c\right)/\hbar}
\end{aligned} \qquad (5.4.2)$$

and for photons (Bose particles) from $(4.5.9)$ for a certain μ we define

$$\begin{aligned}
\boldsymbol{\varphi}^{+}(\underline{x}) &= \frac{1}{\sqrt{\varepsilon_0\hbar}}\int \frac{d^3k}{\sqrt{2\omega_k(2\pi)^3}}\sum_{\lambda=0}^{3}\boldsymbol{a}_\lambda(\vec{k})\,\varepsilon_{\mu,\lambda}(\vec{k})e^{\mathrm{i}\left(\vec{k}\vec{x}-\omega_k x^0/c\right)/\hbar},\\
\boldsymbol{\varphi}^{-}(\underline{x}) &= \frac{1}{\sqrt{\varepsilon_0\hbar}}\int \frac{d^3k}{\sqrt{2\omega_k(2\pi)^3}}\sum_{\lambda=0}^{3}\boldsymbol{a}_\lambda^{\dagger}(\vec{k})\,\varepsilon_{\mu,\lambda}(\vec{k})e^{-\mathrm{i}\left(\vec{k}\vec{x}-\omega_k x^0/c\right)/\hbar}.
\end{aligned} \qquad (5.4.3)$$

The parts $\boldsymbol{\varphi}^{+}(\underline{x})$ contain the wave functions $e^{\mathrm{i}\left(\vec{p}\vec{x}-E(\vec{p})x^0/c\right)/\hbar}$ or $e^{\mathrm{i}\left(\vec{k}\vec{x}-\omega_k x^0/c\right)/\hbar}$ and the annihilation operators $\boldsymbol{b}_r(\vec{p}), \boldsymbol{d}_r(\vec{p})$ or $\boldsymbol{a}_\lambda(\vec{k})$. The parts $\boldsymbol{\varphi}^{-}(\underline{x})$ contain the wave functions $e^{-\mathrm{i}\left(\vec{p}\vec{x}-E(\vec{p})x^0/c\right)/\hbar}$ or $e^{-\mathrm{i}\left(\vec{k}\vec{x}-\omega_k x^0/c\right)/\hbar}$ and the creation operators $\boldsymbol{b}_r^{\dagger}(\vec{p},), \boldsymbol{d}_r^{\dagger}(\vec{p})$ or $\boldsymbol{a}_\lambda^{\dagger}(\vec{k})$.

We give the field operators $\boldsymbol{\varphi}_A(\underline{x})$ and $\boldsymbol{\varphi}_B(\underline{y})$, which are both either bosonic or fermionic. Owing to $(5.4.1)$ the product of these field operators reads

$$\boldsymbol{\varphi}_A(\underline{x})\boldsymbol{\varphi}_B(\underline{y}) = \boldsymbol{\varphi}_A^{+}(\underline{x})\boldsymbol{\varphi}_B^{+}(\underline{y}) + \boldsymbol{\varphi}_A^{+}(\underline{x})\boldsymbol{\varphi}_B^{-}(\underline{y}) + \boldsymbol{\varphi}_A^{-}(\underline{x})\boldsymbol{\varphi}_B^{+}(\underline{y}) + \boldsymbol{\varphi}_A^{-}(\underline{x})\boldsymbol{\varphi}_B^{-}(\underline{y}) . \qquad (5.4.4)$$

The second term on the right hand side contains products of annihilation operators and creation operators, where the annihilation operators, $\boldsymbol{b}_r(\vec{p}), \boldsymbol{d}_r(\vec{p})$ or $\boldsymbol{a}_\lambda(\vec{k})$, are on the left. We apply here the normal ordering of the operators $\boldsymbol{\varphi}^{+}$ and $\boldsymbol{\varphi}^{-}$ (in section 3.3 it has been defined for the operators $\boldsymbol{b}, \boldsymbol{d}, \cdots$ and it was used in section 4.6). If we form the normal ordered product

$$:\boldsymbol{\varphi}_A(\underline{x})\boldsymbol{\varphi}_B(\underline{y}):$$

we have to arrange the partial operators in $(5.4.4)$ so that in no product an operator $\boldsymbol{\varphi}^-$ is at right of $\boldsymbol{\varphi}^+$, i.e. in the second term on the right hand side $\boldsymbol{\varphi}^+(\underline{x})$ and $\boldsymbol{\varphi}^-(\underline{y})$ have to be interchanged. As demanded in section 3.3 for Fermions (Dirac particles) field operators this exchange goes along with a change of the sign, and for bosonic operators the sign remains the same. Therefore we introduce the sign parameter

$$\begin{aligned} &\varepsilon = 1 \quad \text{for bosonic operators} \\ &\varepsilon = -1 \quad \text{for fermionic operators.} \end{aligned} \tag{5.4.5}$$

So we can write the normal ordering of $(5.4.4)$ as follows

$$:\boldsymbol{\varphi}_A(\underline{x})\boldsymbol{\varphi}_B(\underline{y}): = \boldsymbol{\varphi}_A^+(\underline{x})\boldsymbol{\varphi}_B^+(\underline{y}) + \varepsilon\boldsymbol{\varphi}_B^-(\underline{y})\boldsymbol{\varphi}_A^+(\underline{x}) + \boldsymbol{\varphi}_A^-(\underline{x})\boldsymbol{\varphi}_B^+(\underline{y}) + \boldsymbol{\varphi}_A^-(\underline{x})\boldsymbol{\varphi}_B^-(\underline{y}). \tag{5.4.6}$$

Normal ordering of a product of, say, four field operators $\boldsymbol{\varphi}_A(\underline{v}), \boldsymbol{\varphi}_B(\underline{w}), \boldsymbol{\varphi}_C(\underline{x})$ and $\boldsymbol{\varphi}_D(\underline{y})$ creates summands like (for instance)

$$:\boldsymbol{\varphi}_A^+(\underline{v})\boldsymbol{\varphi}_B^-(\underline{w})\boldsymbol{\varphi}_C^+(\underline{x})\boldsymbol{\varphi}_D^-(\underline{y}): = \varepsilon^3\boldsymbol{\varphi}_B^-(\underline{w})\boldsymbol{\varphi}_D^-(\underline{y})\boldsymbol{\varphi}_A^+(\underline{v})\boldsymbol{\varphi}_C^+(\underline{x}). \tag{5.4.7}$$

Thus, generally, in every summand the partial operators $\boldsymbol{\varphi}^+$ and $\boldsymbol{\varphi}^-$ are reordered in such a way that all $\boldsymbol{\varphi}^-$'s stand to the left of the $\boldsymbol{\varphi}^+$'s. Every transposition of $\boldsymbol{\varphi}^+$ and $\boldsymbol{\varphi}^-$ adds a factor ε to the summand.

Using $(5.4.5)$ we interchange the operators in two summands of $(5.4.6)$. From $(5.4.2)$, $(3.3.9)$,$(3.3.27)$ and $(4.5.12)$ follows

$$\begin{aligned} &\boldsymbol{\varphi}_A^+(\underline{x})\boldsymbol{\varphi}_B^+(\underline{y}) = \varepsilon\boldsymbol{\varphi}_B^+(\underline{y})\boldsymbol{\varphi}_A^+(\underline{x}) \quad \text{and} \\ &\boldsymbol{\varphi}_A^-(\underline{x})\boldsymbol{\varphi}_B^-(\underline{y}) = \varepsilon\boldsymbol{\varphi}_B^-(\underline{y})\boldsymbol{\varphi}_A^-(\underline{x}), \end{aligned} \tag{5.4.8}$$

which we insert in $(5.4.6)$ as follows

$$\begin{aligned} &:\boldsymbol{\varphi}_A(\underline{x})\boldsymbol{\varphi}_B(y): \\ &= \varepsilon\boldsymbol{\varphi}_B^+(\underline{y})\boldsymbol{\varphi}_A^+(\underline{x}) + \varepsilon\boldsymbol{\varphi}_B^-(\underline{y})\boldsymbol{\varphi}_A^+(\underline{x}) + \boldsymbol{\varphi}_A^-(\underline{x})\boldsymbol{\varphi}_B^+(\underline{y}) + \varepsilon\boldsymbol{\varphi}_B^-(\underline{y})\boldsymbol{\varphi}_A^-(\underline{x}) \\ &= \varepsilon\left(\boldsymbol{\varphi}_B^+(\underline{y})\boldsymbol{\varphi}_A^+(\underline{x}) + \boldsymbol{\varphi}_B^-(\underline{y})\boldsymbol{\varphi}_A^+(\underline{x}) + \varepsilon\boldsymbol{\varphi}_A^-(\underline{x})\boldsymbol{\varphi}_B^+(\underline{y}) + \boldsymbol{\varphi}_B^-(\underline{y})\boldsymbol{\varphi}_A^-(\underline{x})\right) \\ &= \varepsilon:\boldsymbol{\varphi}_B(y)\boldsymbol{\varphi}_A(\underline{x}):. \end{aligned} \tag{5.4.9}$$

Thus, except for a possible sign factor the order of the operators within the argument of a normal product does not matter. This holds generally for normal products involving more than two factors.

A similar rule holds for time ordering. Due to $(3.4.2)$ and $(4.7.2)$ we write

$$\begin{aligned}
&\text{for } t_x > t_y \quad T\left(\boldsymbol{\varphi}_A(\underline{x})\boldsymbol{\varphi}_B(\underline{y})\right) = \boldsymbol{\varphi}_A(\underline{x})\boldsymbol{\varphi}_B(\underline{y}) \\
&\qquad\qquad T\left(\boldsymbol{\varphi}_B(\underline{y})\boldsymbol{\varphi}_A(\underline{x})\right) = \varepsilon\boldsymbol{\varphi}_A(\underline{x})\boldsymbol{\varphi}_B(\underline{y}) = \varepsilon T\left(\boldsymbol{\varphi}_A(\underline{x})\boldsymbol{\varphi}_B(\underline{y})\right), \\
&\text{for } t_y > t_x \quad T\left(\boldsymbol{\varphi}_A(\underline{x})\boldsymbol{\varphi}_B(\underline{y})\right) = \varepsilon\boldsymbol{\varphi}_B(\underline{y})\boldsymbol{\varphi}_A(\underline{x}) \\
&\qquad\qquad T\left(\boldsymbol{\varphi}_B(\underline{y})\boldsymbol{\varphi}_A(\underline{x})\right) = \boldsymbol{\varphi}_B(\underline{y})\boldsymbol{\varphi}_A(\underline{x}) = \varepsilon T\left(\boldsymbol{\varphi}_A(\underline{x})\boldsymbol{\varphi}_B(\underline{y})\right).
\end{aligned} \tag{5.4.10}$$

We note that the time ordered product is also invariant under permutation except for a sign factor.

Now we evaluate time-ordered products and start with the product $T\left(\boldsymbol{\varphi}_A(\underline{x})\boldsymbol{\varphi}_B(\underline{y})\right)$ of two field operators. First we assume that the time arguments are ordered such that $t_x > t_y$, allowing us to write

$$\begin{aligned}
&T\left(\boldsymbol{\varphi}_A(\underline{x})\boldsymbol{\varphi}_B(y)\right) = \boldsymbol{\varphi}_A(\underline{x})\boldsymbol{\varphi}_B(y) \\
&= \boldsymbol{\varphi}_A^+(\underline{x})\boldsymbol{\varphi}_B^+(\underline{y}) + \boldsymbol{\varphi}_A^-(\underline{x})\boldsymbol{\varphi}_B^-(\underline{y}) + \boldsymbol{\varphi}_A^-(\underline{x})\boldsymbol{\varphi}_B^+(\underline{y}) + \boldsymbol{\varphi}_A^+(\underline{x})\boldsymbol{\varphi}_B^-(\underline{y}).
\end{aligned} \tag{5.4.11}$$

Except for the last term the products are already normal ordered. This term can be written as

$$\begin{aligned}
\boldsymbol{\varphi}_A^+(\underline{x})\boldsymbol{\varphi}_B^-(\underline{y}) &= \pm\boldsymbol{\varphi}_B^-(\underline{y})\boldsymbol{\varphi}_A^+(\underline{x}) + \boldsymbol{\varphi}_A^+(\underline{x})\boldsymbol{\varphi}_B^-(\underline{y}) \mp \boldsymbol{\varphi}_B^-(\underline{y})\boldsymbol{\varphi}_A^+(\underline{x}) \\
&= \pm\boldsymbol{\varphi}_B^-(\underline{y})\boldsymbol{\varphi}_A^+(\underline{x}) + \left[\boldsymbol{\varphi}_A^+(\underline{x}),\boldsymbol{\varphi}_B^-(\underline{y})\right]_\mp \quad ,
\end{aligned} \tag{5.4.12}$$

where we assign the upper sign to Bosons and the lower one to Fermions. Therefore, with $(5.4.5)$ we obtain

$$\boldsymbol{\varphi}_A^+(\underline{x})\boldsymbol{\varphi}_B^-(\underline{y}) = \varepsilon\boldsymbol{\varphi}_B^-(\underline{y})\boldsymbol{\varphi}_A^+(\underline{x}) + \left[\boldsymbol{\varphi}_A^+(\underline{x}),\boldsymbol{\varphi}_B^-(\underline{y})\right]_\mp . \tag{5.4.13}$$

Due to $(5.4.3)$ the Boson commutator

$$\left[\boldsymbol{\varphi}_A^+(\underline{x}),\boldsymbol{\varphi}_B^-(\underline{y})\right]_- \tag{5.4.14}$$

is a linear combination of the commutators $\left[\boldsymbol{a}_\lambda\left(\vec{k}\right),\boldsymbol{a}_{\lambda'}^\dagger\left(\vec{k}'\right)\right]_-$. According to $(4.5.11)$ they equal $-g_{\lambda\lambda'}\delta^3\left(\vec{k}-\vec{k}'\right)$ and therefore the commutator $(5.4.14)$ is not an operator but a c-number (real or complex). Due to $(5.4.2)$ the Fermion commutator

$$\left[\boldsymbol{\varphi}_A^+(\underline{x}),\boldsymbol{\varphi}_B^-(\underline{y})\right]_+ \tag{5.4.15}$$

is even zero because it contains anticommutators of the form $\left[\boldsymbol{b}_r\left(\vec{p}\right),\boldsymbol{d}_r^\dagger\left(\vec{p}'\right)\right]_+$, which vanish (c.f. $(3.3.27)$). For a c-number c the relation $\langle 0|c|0\rangle = c$ holds, i.e. it can be replaced by its vacuum expectation value and we can write

$$\left[\boldsymbol{\varphi}_A^+(\underline{x}),\boldsymbol{\varphi}_B^-(\underline{y})\right]_\mp = \langle 0|\left[\boldsymbol{\varphi}_A^+(\underline{x}),\boldsymbol{\varphi}_B^-(\underline{y})\right]_\mp|0\rangle$$
$$= \langle 0|\boldsymbol{\varphi}_A^+(\underline{x})\boldsymbol{\varphi}_B^-(\underline{y})|0\rangle \mp \langle 0|\boldsymbol{\varphi}_B^-(\underline{y})\boldsymbol{\varphi}_A^+(\underline{x})|0\rangle. \tag{5.4.16}$$

The last term in $(5.4.16)$ vanishes because $\boldsymbol{\varphi}_A^+(\underline{x})$ contains annihilation operators which act on $|0\rangle$. Because of

$$\boldsymbol{\varphi}^+|0\rangle = \langle 0|\boldsymbol{\varphi}^- = 0 \tag{5.4.17}$$

we have

$$\langle 0|\boldsymbol{\varphi}_A(\underline{x})\boldsymbol{\varphi}_B(\underline{y})|0\rangle = \langle 0|\boldsymbol{\varphi}_A^+(\underline{x})\boldsymbol{\varphi}_B^+(\underline{y})|0\rangle + \langle 0|\boldsymbol{\varphi}_A^+(\underline{x})\boldsymbol{\varphi}_B^-(\underline{y})|0\rangle$$
$$+ \langle 0|\boldsymbol{\varphi}_A^-(\underline{x})\boldsymbol{\varphi}_B^+(\underline{y})|0\rangle + \langle 0|\boldsymbol{\varphi}_A^-(\underline{x})\boldsymbol{\varphi}_B^-(\underline{y})|0\rangle \tag{5.4.18}$$
$$= \langle 0|\boldsymbol{\varphi}_A^+(\underline{x})\boldsymbol{\varphi}_B^-(\underline{y})|0\rangle,$$

which we insert in $(5.4.16)$ like this

$$\left[\boldsymbol{\varphi}_A^+(\underline{x}),\boldsymbol{\varphi}_B^-(\underline{y})\right]_\mp = \langle 0|\boldsymbol{\varphi}_A(\underline{x}),\boldsymbol{\varphi}_B(\underline{y})|0\rangle = \langle 0|T\left(\boldsymbol{\varphi}_A(\underline{x}),\boldsymbol{\varphi}_B(\underline{y})\right)|0\rangle. \tag{5.4.19}$$

The last step is based on the condition $t_x > t_y$ chosen preceding $(5.4.11)$. We insert $(5.4.19)$ in $(5.4.13)$ and put this expression in $(5.4.11)$, which yields

$$T\left(\boldsymbol{\varphi}_A(\underline{x})\boldsymbol{\varphi}_B(\underline{y})\right) = :\boldsymbol{\varphi}_A(\underline{x})\boldsymbol{\varphi}_B(\underline{y}):+\langle 0|T\left(\boldsymbol{\varphi}_A(\underline{x})\boldsymbol{\varphi}_B(\underline{y})\right)|0\rangle. \tag{5.4.20}$$

This equation has been derived for $t_x > t_y$. However, it also holds true in the opposite case $t_y > t_x$. We show this using $(5.4.9)$, $(5.4.10)$ and $(5.4.20)$

$$T\left(\boldsymbol{\varphi}_A(\underline{x})\boldsymbol{\varphi}_B(\underline{y})\right) = \varepsilon T\left(\boldsymbol{\varphi}_B(\underline{y})\boldsymbol{\varphi}_A(\underline{x})\right)$$
$$= \varepsilon :\boldsymbol{\varphi}_B(\underline{y})\boldsymbol{\varphi}_A(\underline{x}):+\varepsilon\langle 0|T\left(\boldsymbol{\varphi}_B(\underline{y})\boldsymbol{\varphi}_A(\underline{x})\right)|0\rangle \tag{5.4.21}$$
$$= :\boldsymbol{\varphi}_A(\underline{x})\boldsymbol{\varphi}_B(\underline{y}):+\langle 0|T\left(\boldsymbol{\varphi}_A(\underline{x})\boldsymbol{\varphi}_B(\underline{y})\right)|0\rangle.$$

Thus, the relation $(5.4.20)$ is valid in general. The last term in $(5.4.21)$ is a vacuum expectation value and therefore a c-number. Further, the normal ordering of a c-number is defined as the c-number itself. Therefore, “normal ordering” of this term in $(5.4.21)$ does no harm and usually this equation is written as

$$T\left(\boldsymbol{\varphi}_A(\underline{x})\boldsymbol{\varphi}_B(\underline{y})\right) = :\boldsymbol{\varphi}_A(\underline{x})\boldsymbol{\varphi}_B(\underline{y}):+:\langle 0|T\left(\boldsymbol{\varphi}_A(\underline{x})\boldsymbol{\varphi}_B(\underline{y})\right)|0\rangle:. \tag{5.4.22}$$

Since the vacuum expectation value of a *T*-product of two operators frequently occurs it is given a compact notion. One defines

$$\underbracket{\boldsymbol{\varphi}_A(\underline{x})\boldsymbol{\varphi}_B}(\underline{y}) \equiv \langle 0|T\left(\boldsymbol{\varphi}_A(\underline{x})\boldsymbol{\varphi}_B(\underline{y})\right)|0\rangle. \tag{5.4.23}$$

It is given the name "time ordered contraction" (or simply contraction) of two operators. By means of $(5.4.23)$ the equation $(5.4.22)$ is written like this

$$T\left(\boldsymbol{\varphi}_A(\underline{x})\boldsymbol{\varphi}_B(\underline{y})\right) = :\boldsymbol{\varphi}_A(\underline{x})\boldsymbol{\varphi}_B(\underline{y}): + :\underbracket{\boldsymbol{\varphi}_A(\underline{x})\boldsymbol{\varphi}_B(\underline{y})}: . \tag{5.4.24}$$

Multiple contraction can be applied to a product of many operators which we name $\boldsymbol{A,B,C,\cdots,M}$ and which we give in normal ordering. For example, using $(5.4.9)$ the following contraction can be written like this

$$:\boldsymbol{ABCD\,EF\cdots KLM\cdots}: = \varepsilon_p :\boldsymbol{ABF\cdots KM\cdots}: \underbracket{\boldsymbol{C\,E}}\,\underbracket{\boldsymbol{D\,L}} . \tag{5.4.25}$$

here $\varepsilon_p = \pm 1$ is the parity of the permutation of (fermionic) operators during the moving to the right.

The procedure which leads us to $(5.4.24)$ can be extended to the product of three operators. We first assume the time ordering $t_x > t_y > t_z$. Then we find using $(5.4.20)$ and $(5.4.23)$

$$\begin{aligned} T\left(\boldsymbol{\varphi}_A(\underline{x})\boldsymbol{\varphi}_B(\underline{y})\boldsymbol{\varphi}_C(\underline{z})\right) &= T\left(\boldsymbol{\varphi}_A(\underline{x})\boldsymbol{\varphi}_B(\underline{y})\right)\boldsymbol{\varphi}_C(\underline{z}) \\ &= :\boldsymbol{\varphi}_A(\underline{x})\boldsymbol{\varphi}_B(\underline{y}):\boldsymbol{\varphi}_C(\underline{z}) + \underbracket{\boldsymbol{\varphi}_A(\underline{x})\boldsymbol{\varphi}_B(\underline{y})}\boldsymbol{\varphi}_C(\underline{z}). \end{aligned} \tag{5.4.26}$$

We deal with the first term on the right hand side and drop the arguments $\underline{x}, \underline{y}$, and $\underline{z}$:

$$\begin{aligned} &:\boldsymbol{\varphi}_A(\underline{x})\boldsymbol{\varphi}_B(\underline{y}):\boldsymbol{\varphi}_C(\underline{z}) \\ &= \left(\boldsymbol{\varphi}_A^-\boldsymbol{\varphi}_B^- + \boldsymbol{\varphi}_A^-\boldsymbol{\varphi}_B^+ + \boldsymbol{\varphi}_A^+\boldsymbol{\varphi}_B^+ + \varepsilon_{AB}\boldsymbol{\varphi}_B^-\boldsymbol{\varphi}_A^+\right)\cdot\left(\boldsymbol{\varphi}_C^- + \boldsymbol{\varphi}_C^+\right) \\ &= \boldsymbol{\varphi}_A^-\boldsymbol{\varphi}_B^-\boldsymbol{\varphi}_C^- + \boldsymbol{\varphi}_A^-\boldsymbol{\varphi}_B^+\boldsymbol{\varphi}_C^- + \boldsymbol{\varphi}_A^+\boldsymbol{\varphi}_B^+\boldsymbol{\varphi}_C^- + \varepsilon_{AB}\boldsymbol{\varphi}_B^-\boldsymbol{\varphi}_A^+\boldsymbol{\varphi}_C^- \\ &+ \boldsymbol{\varphi}_A^-\boldsymbol{\varphi}_B^-\boldsymbol{\varphi}_C^+ + \boldsymbol{\varphi}_A^-\boldsymbol{\varphi}_B^+\boldsymbol{\varphi}_C^+ + \boldsymbol{\varphi}_A^+\boldsymbol{\varphi}_B^+\boldsymbol{\varphi}_C^+ + \varepsilon_{AB}\boldsymbol{\varphi}_B^-\boldsymbol{\varphi}_A^+\boldsymbol{\varphi}_C^+. \end{aligned} \tag{5.4.27}$$

We work on the second up to the fourth term on the right hand side, where we move $\boldsymbol{\varphi}_C^-$ to the left. By means of $(5.4.13)$ we write

$$\boldsymbol{\varphi}_A^-\boldsymbol{\varphi}_B^+\boldsymbol{\varphi}_C^- = \varepsilon_{BC}\boldsymbol{\varphi}_A^-\boldsymbol{\varphi}_C^-\boldsymbol{\varphi}_B^+ + \boldsymbol{\varphi}_A^-\left[\boldsymbol{\varphi}_B^+,\boldsymbol{\varphi}_C^-\right]_\mp , \tag{5.4.28}$$

$$\begin{aligned} \boldsymbol{\varphi}_A^+\boldsymbol{\varphi}_B^+\boldsymbol{\varphi}_C^- &= \boldsymbol{\varphi}_A^+\left(\varepsilon_{BC}\boldsymbol{\varphi}_C^-\boldsymbol{\varphi}_B^+ + \left[\boldsymbol{\varphi}_B^+,\boldsymbol{\varphi}_C^-\right]_\mp\right) \\ &= \varepsilon_{BC}\varepsilon_{AC}\boldsymbol{\varphi}_C^-\boldsymbol{\varphi}_A^+\boldsymbol{\varphi}_B^+ + \varepsilon_{BC}\left[\boldsymbol{\varphi}_A^+,\boldsymbol{\varphi}_C^-\right]_\mp\boldsymbol{\varphi}_B^+ + \boldsymbol{\varphi}_A^+\left[\boldsymbol{\varphi}_B^+,\boldsymbol{\varphi}_C^-\right]_\mp , \end{aligned} \tag{5.4.29}$$

$$\varepsilon_{AB}\boldsymbol{\varphi}_B^-\boldsymbol{\varphi}_A^+\boldsymbol{\varphi}_C^- = \varepsilon_{AB}\varepsilon_{AC}\boldsymbol{\varphi}_B^-\boldsymbol{\varphi}_C^-\boldsymbol{\varphi}_A^+ + \varepsilon_{AC}\boldsymbol{\varphi}_B^-\left[\boldsymbol{\varphi}_A^+,\boldsymbol{\varphi}_C^-\right]_\mp . \tag{5.4.30}$$

We insert $(5.4.28)$ up to $(5.4.30)$ in $(5.4.27)$ like this

$$
\begin{aligned}
:\varphi_A\varphi_B:\varphi_C =& \varphi_A^-\varphi_B^-\varphi_C^- + \varepsilon_{BC}\varphi_A^-\varphi_C^-\varphi_B^+ + \varepsilon_{BC}\varepsilon_{AC}\varphi_C^-\varphi_A^+\varphi_B^+ \\
&+ \varepsilon_{AB}\varepsilon_{AC}\varphi_B^-\varphi_C^-\varphi_A^+ + \varphi_A^-\varphi_B^-\varphi_C^+ + \varphi_A^-\varphi_B^+\varphi_C^+ \\
&+ \varphi_A^+\varphi_B^+\varphi_C^+ + \varepsilon_{AB}\varphi_B^-\varphi_A^+\varphi_C^+ + \varphi_A^-\left[\varphi_B^+,\varphi_C^-\right]_\mp \\
&+ \varepsilon_{BC}\left[\varphi_A^+,\varphi_C^-\right]_\mp\varphi_B^+ + \varphi_A^+\left[\varphi_B^+,\varphi_C^-\right]_\mp + \varepsilon_{AB}\varphi_B^-\left[\varphi_A^+,\varphi_C^-\right]_\mp .
\end{aligned}
\tag{5.4.31}
$$

Due to $(5.4.16)$ and $(5.4.18)$ we can write

$$
\left[\varphi_A^+,\varphi_B^-\right]_\mp = \langle 0|\varphi_A^+\varphi_B^-|0\rangle = \langle 0|\varphi_A\varphi_B|0\rangle . \tag{5.4.32}
$$

Analogously we obtain according to $(5.4.23)$

$$
\left[\varphi_A^+,\varphi_C^-\right]_\mp = \langle 0|\varphi_A\varphi_C|0\rangle = \underbrace{\varphi_A\varphi_C} . \tag{5.4.33}
$$

We remember

$$
\begin{aligned}
:\varphi_A\varphi_B\varphi_C: =& \varphi_A^-\varphi_B^-\varphi_C^- + \varphi_A^-\varphi_B^-\varphi_C^+ + :\varphi_A^-\varphi_B^+\varphi_C^-: + \varphi_A^-\varphi_B^+\varphi_C^+ \\
&+ :\varphi_A^+\varphi_B^-\varphi_C^-: + :\varphi_A^+\varphi_B^-\varphi_C^+: + :\varphi_A^+\varphi_B^+\varphi_C^-: + \varphi_A^+\varphi_B^+\varphi_C^+ \\
=& \varphi_A^-\varphi_B^-\varphi_C^- + \varphi_A^-\varphi_B^-\varphi_C^+ + \varepsilon_{BC}\varphi_A^-\varphi_C^-\varphi_B^+ + \varphi_A^-\varphi_B^+\varphi_C^+ \\
&+ \varepsilon_{AB}\varepsilon_{AC}\varphi_B^-\varphi_C^-\varphi_A^+ + \varepsilon_{AB}\varphi_B^-\varphi_A^+\varphi_C^+ + \varepsilon_{BC}\varepsilon_{AC}\varphi_C^-\varphi_A^+\varphi_B^+ + \varphi_A^+\varphi_B^+\varphi_C^+ .
\end{aligned}
\tag{5.4.34}
$$

We insert $(5.4.33)$ and $(5.4.34)$ in $(5.4.31)$ as follows

$$
:\varphi_A\varphi_B:\varphi_C = :\varphi_A\varphi_B\varphi_C: + \varphi_A\underbrace{\varphi_B\varphi_C} + \underbrace{\varphi_A\varphi_B\varphi_C} . \tag{5.4.35}
$$

The last term has no sign factor because according to $(5.4.25)$ and $(5.4.33)$ the following is true

$$
\begin{aligned}
&\varepsilon_{BC}\left[\varphi_A^+,\varphi_C^-\right]_\mp\varphi_B^+ = \underbrace{\varphi_A^+\varphi_B^+\varphi_C^-} = \underbrace{\varphi_A\varphi_B^+\varphi_C} \\
&\text{and } \varepsilon_{AB}\varphi_B^-\left[\varphi_A^+,\varphi_C^-\right]_\mp = \underbrace{\varphi_A^+\varphi_B^-\varphi_C^-} = \underbrace{\varphi_A\varphi_B^-\varphi_C} .
\end{aligned}
\tag{5.4.36}
$$

We insert $(5.4.35)$ in $(5.4.26)$ like this

$$
\begin{aligned}
&T\left(\varphi_A(\underline{x})\varphi_B(\underline{y})\varphi_C(\underline{z})\right) \\
&= :\varphi_A(\underline{x})\varphi_B(\underline{y})\varphi_C(\underline{z}): + :\underbrace{\varphi_A(\underline{x})\varphi_B}(\underline{y})\varphi_C(\underline{z}): \\
&+ :\underbrace{\varphi_A(\underline{x})\varphi_B(\underline{y})\varphi_C}(\underline{z}): + :\varphi_A(\underline{x})\underbrace{\varphi_B(\underline{y})\varphi_C}(\underline{z}): .
\end{aligned}
\tag{5.4.37}
$$

this result has been derived under the assumption $t_x, t_y > t_z$. One can show that $(5.4.37)$ holds true for the cases $t_x, t_z > t_y$ and $t_y, t_z > t_x$ as Greiner and Reinhardt, 1996, p. 229, mentioned.

The relation $(5.4.37)$ can be generalized to products of more operators. This is the essence of Wick's theorem. Given the operators $\boldsymbol{A}, \boldsymbol{B}, \cdots, \boldsymbol{Z}$ one can write the time ordered product like this

$$
\begin{aligned}
&T\left(\boldsymbol{ABC}\cdots\boldsymbol{XYZ}\right) = :\boldsymbol{ABC}\cdots\boldsymbol{XYZ}: \\
&+:\underbrace{\boldsymbol{AB}}\boldsymbol{C}\cdots\boldsymbol{XYZ}:+:\underbrace{\boldsymbol{A}\boldsymbol{B}\boldsymbol{C}}\cdots\boldsymbol{XYZ}:+\cdots+:\boldsymbol{ABC}\cdots\boldsymbol{X}\underbrace{\boldsymbol{YZ}}: \\
&+:\underbrace{\boldsymbol{AB}}\,\underbrace{\boldsymbol{CD}}\cdots\boldsymbol{XYZ}:+:\underbrace{\boldsymbol{ABCD}}\cdots\boldsymbol{XYZ}:+\cdots+:\boldsymbol{ABC}\cdots\underbrace{\boldsymbol{WX}}\,\underbrace{\boldsymbol{YZ}}:
\end{aligned}
\tag{5.4.38}
$$

+ sum over triply contracted terms

+ higher contractions.

The second line of $(5.4.38)$ contains all possible single contractions, the third line all double contractions and so on. We recognize that our earlier results $(5.4.24)$ and $(5.4.37)$ are special cases of $(5.4.38)$. Greiner and Reinhardt. 1996, p. 231, have given a general proof of Wick's theorem.

We see, contractions play an important role in Wick's theorem. We put together the contractions with which we have become acquainted. First, we write down the contractions of Dirac fields. Equations $(5.4.23)$ and $(3.4.4)$ reveal

$$
\underbrace{\boldsymbol{\varphi}_\alpha(\underline{x})\overline{\boldsymbol{\varphi}}_\beta}(\underline{y}) = \langle 0|T\left(\boldsymbol{\varphi}_\alpha(\underline{x})\overline{\boldsymbol{\varphi}}_\beta(\underline{y})\right)|0\rangle = \mathrm{i}S_{F,\alpha\beta}\left(\underline{x}-\underline{y}\right) \tag{5.4.39}
$$

and $(3.4.38)$ states

$$
\begin{aligned}
S_{F,\alpha\beta}\left(\underline{x}-\underline{y}\right) &= \left(\mathrm{i}\underline{\underline{\nabla\!\!\!/}}_x + mc\underline{\underline{1}}/\hbar\right)_{\alpha\beta}\Delta_F\left(\underline{x}-\underline{y}\right) \\
&= \int \frac{d^4q}{(2\pi)^4} e^{-\mathrm{i}\underline{q}(\underline{x}-\underline{y})} \cdot \frac{\left(\underline{\underline{q\!\!\!/}} + mc\underline{\underline{1}}/\hbar\right)_{\alpha\beta}}{\underline{q}^2 - (mc/\hbar)^2 + \mathrm{i}\varepsilon}.
\end{aligned}
\tag{5.4.40}
$$

For photon fields the contraction reads due to $(5.4.23)$, $(4.7.1)$ and $(4.7.9)$.

$$
\begin{aligned}
\underbrace{\boldsymbol{A}^{(\mu)}(\underline{x})\boldsymbol{A}^{(\nu)}}(\underline{y}) &= \langle 0|T\left(\boldsymbol{A}^{(\mu)}(\underline{x})\boldsymbol{A}^{(\nu)}(\underline{y})\right)|0\rangle = \mathrm{i}D_F^{\mu\nu}\left(\underline{x}-\underline{y}\right) \\
&= -\mathrm{i}g_{\mu\nu}\int \frac{d^4q}{(2\pi)^4} \cdot \frac{e^{-\mathrm{i}\underline{q}(\underline{x}-\underline{y})}}{\underline{q}^2 + \mathrm{i}\varepsilon} \cdot \frac{\hbar}{\varepsilon_0 c}.
\end{aligned}
\tag{5.4.41}
$$

5.5 Interaction between quantized Dirac- and Maxwell fields

The classical Lagrangian of quantum electrodynamics involving a Dirac particle and a photon field was introduced in $(4.3.1)$ up to $(4.3.3)$. We write it like this

$$
\mathcal{L} = \mathcal{L}_{\text{Dirac, free}} + \mathcal{L}_{\text{Maxwel,free}} + \mathcal{L}_1 \tag{5.5.1}
$$

with

$$\mathcal{L}_{\text{Dirac,free}} = \underline{\psi}^{\dagger}\left(\mathrm{i}\hbar\dot{\underline{\psi}} + \mathrm{i}\sum_{i=1}^{3}\hbar c\underline{\underline{\alpha}}_i\frac{d\underline{\psi}}{dx^{(i)}} - mc^2\underline{\underline{\beta}}\underline{\psi}\right),$$

$$\mathcal{L}_{\text{Maxwell, free}} = \frac{-1}{2\mu_0}\sum_{\mu,\nu=0}^{3}\frac{dA_{\nu}}{dx^{(\mu)}}\cdot\frac{dA^{(\nu)}}{dx_{\mu}}, \tag{5.5.2}$$

$$\mathcal{L}_1 = -\sum_{\mu=0}^{3} g_{\mu\mu} ce\underline{\psi}^{\dagger}(\vec{x},t)\underline{\underline{\alpha}}_{\mu}\underline{\psi}(\vec{x},t)A^{(\mu)}.$$

Due to $(1.4.13)$, $(3.1.14)$, $(4.2.9)$ and $(5.5.1)$ the Hamilton density of the system reads

$$\mathcal{H} = \sum_{k=1}^{4}\frac{\partial\mathcal{L}}{\partial\left(\frac{d\psi_k}{dt}\right)}\frac{d\psi_k}{dt} + \sum_{k=1}^{4}\frac{\partial\mathcal{L}}{\partial\left(\frac{d\psi_k^{\dagger}}{dt}\right)}\frac{d\psi_k^{\dagger}}{dt} - \mathcal{L}_{\text{Dirac. free}}$$

$$+\sum_{\nu=0}^{3}\frac{\partial\mathcal{L}}{\partial\left(\frac{dA^{(\nu)}}{dx^0}\right)}\frac{dA^{(\nu)}}{dx^0} - \mathcal{L}_{\text{Maxwell, free}} \tag{5.5.3}$$

$$+\sum_{\mu=0}^{3} g_{\mu\mu} ec\underline{\psi}^{\dagger}(\vec{x},t)\,\underline{\underline{\alpha}}_{\mu}\underline{\psi}(\vec{x},t)A^{(\mu)}.$$

Line one and line two on the right hand side represent free fields. Therefore, the last line is the perturbation part of the Hamilton density according to $(5.1.1)$

$$\mathcal{H}_1 = \sum_{\mu=0}^{3} g_{\mu\mu} ec\underline{\psi}^{\dagger}(\vec{x},t)\,\underline{\underline{\alpha}}_{\mu}\underline{\psi}(\vec{x},t)A^{(\mu)}(\vec{x},t). \tag{5.5.4}$$

In line with $(3.3.1)$, $(3.3.2)$ and $(4.5.4)$ we replace $\underline{\psi},\underline{\psi}^{\dagger}$ and $A^{(\mu)}$ by field operators and apply normal ordering in order to renormalize the energy scale (c.f. $(3.3.32)$)

$$\boldsymbol{\mathcal{H}}_1(\underline{x}) = \sum_{\mu=0}^{3} g_{\mu\mu} ec{:}\underline{\boldsymbol{\psi}}^{\dagger}(\underline{x})\underline{\underline{\alpha}}_{\mu}\underline{\boldsymbol{\psi}}(\underline{x})\boldsymbol{A}^{(\mu)}(\underline{x}){:}. \tag{5.5.5}$$

We insert $(5.5.5)$ in $(5.3.8)$ to obtain the $\boldsymbol{S}$ operator for interactions between Dirac particles and photons

$$\boldsymbol{S} = \boldsymbol{1} + \sum_{n=1}^{\infty}\frac{1}{n!}(-\mathrm{i}ec)^n\int\frac{d^4x_1}{c}\cdots\frac{d^4x_n}{c}$$
$$\cdot T\left({:}\sum_{\mu_1=0}^{3}\frac{g_{\mu_1\mu_1}}{\hbar}\underline{\boldsymbol{\psi}}^{\dagger}(\underline{x}_1)\underline{\underline{\alpha}}_{\mu_1}\underline{\boldsymbol{\psi}}(x_1)\boldsymbol{A}^{(\mu_1)}(\underline{x}_1){:}\cdots{:}\sum_{\mu_n=0}^{3}\frac{g_{\mu_n\mu_n}}{\hbar}\underline{\boldsymbol{\psi}}^{\dagger}(\underline{x}_n)\underline{\underline{\alpha}}_{\mu_n}\underline{\boldsymbol{\psi}}(\underline{x}_n)\boldsymbol{A}^{(\mu_n)}(\underline{x}_n){:}\right) \tag{5.5.6}$$

With this infinite series we perform a perturbation approach using the terms of the lowest order (lowest n values). It can be shown that the case $n=1$ is too simple for describing a physical process. The corresponding $\boldsymbol{S}$ operator reads

$\boldsymbol{S}^{(1)} = -\mathrm{ie}\int d^4x \frac{1}{\hbar}\sum_{\mu=0}^{3} g_{\mu\mu} :\underline{\boldsymbol{\psi}}^{\dagger}(\underline{x})\underline{\underline{\alpha}}_{\mu}\underline{\boldsymbol{\psi}}(\underline{x})\boldsymbol{A}^{(\mu)}(\underline{x}):$. Inserting $(3.3.9)$, $(4.5.9)$ and $(3.2.25)$ results in expressions like $\int d^4x\, e^{\mathrm{i}(\pm \underline{p} \pm \underline{p}' \pm \underline{k})\underline{x}/\hbar} = (2\pi)^4 \hbar^4 \delta^4(\pm\underline{p} \pm \underline{p}' \pm \underline{k})$, which stands for $\pm\underline{p} \pm \underline{p}' = \mp\underline{k}'$ or $\pm\vec{p} \pm \vec{p}' = \mp\vec{k}$ and simultatneously for $\pm\sqrt{m^2c^2 + \vec{p}^2} \pm \sqrt{m^2c^2 + \vec{p}'^2} \equiv \pm\frac{E}{c} \pm \frac{E'}{c} = \mp\frac{\omega_k}{c} = \mp\sqrt{\vec{k}^2}$, which is not satisfied simultaneously. Therefore $\boldsymbol{S}^{(1)}$ does not exist.

We go to the second order $\boldsymbol{S}$ operator

$$\boldsymbol{S}^{(2)} = \frac{1}{2!}(-\mathrm{ie})^2 \int d^4x_1 \int d^4x_2$$
$$\cdot T\left(:\sum_{\mu_1=0}^{3} \frac{g_{\mu_1\mu_1}}{\hbar} \underline{\boldsymbol{\psi}}^{\dagger}(\underline{x}_1)\underline{\underline{\alpha}}_{\mu_1}\underline{\boldsymbol{\psi}}(\underline{x}_1)\boldsymbol{A}^{(\mu_1)}(\underline{x}_1)::\sum_{\mu_2=0}^{3} \frac{g_{\mu_2\mu_2}}{\hbar} \underline{\boldsymbol{\psi}}^{\dagger}(\underline{x}_2)\underline{\underline{\alpha}}_{\mu_2}\underline{\boldsymbol{\psi}}(\underline{x}_2)\boldsymbol{A}^{(\mu_2)}(\underline{x}_2): \right) \tag{5.5.7}$$

where $\boldsymbol{S} \simeq \boldsymbol{1} + \boldsymbol{S}^{(2)}$ holds. (5.5.8)

Because all resulting terms in Wick's theorem (see $(5.4.38)$) have normal ordering, the normal ordering in $(5.5.7)$ has no additional consequences and we can apply Wick's theorem as if there were no partial normal ordering in $(5.5.7)$.

Following $(5.4.38)$, in $(5.5.7)$ contractions like $\underbrace{\boldsymbol{\psi}_{\alpha}^{\dagger}(\underline{x})\boldsymbol{\psi}}_{}{}_{\beta}(\underline{x})$ should appear. However, normal ordering generates terms with operator products $\boldsymbol{d}^{\dagger}\boldsymbol{b}, \boldsymbol{d}^{\dagger}\boldsymbol{d}, \boldsymbol{b}^{\dagger}\boldsymbol{b}$ and $\boldsymbol{b}^{\dagger}\boldsymbol{d}$, the vacuum expectation values of which, $\langle 0|\boldsymbol{d}^{\dagger}\boldsymbol{b}|0\rangle$, $\langle 0|\boldsymbol{d}^{\dagger}\boldsymbol{d}|0\rangle$, etc., all vanish, i.e.

$$\underbrace{\boldsymbol{\psi}_{\alpha}^{\dagger}(\underline{x})\boldsymbol{\psi}}_{}{}_{\beta}(\underline{x}) = 0 \tag{5.5.9}$$

Thus, such contractions don't enter in the $\boldsymbol{S}^{(2)}$ operator. Moreover, contractions of the type $\underbrace{\boldsymbol{\psi}_{\alpha}(\underline{x}_1)\boldsymbol{\psi}}_{}{}_{\beta}(\underline{x}_2)$ don't arise, which can be seen using $(5.4.23)$, $(5.4.18)$ and $(5.4.2)$ assuming $t_1 > t_2$

$$\underbrace{\boldsymbol{\psi}_{\alpha}(\underline{x}_1)\boldsymbol{\psi}}_{}{}_{\beta}(\underline{x}_2) = \langle 0|T(\boldsymbol{\psi}_{\alpha}(\underline{x}_1)\boldsymbol{\psi}_{\beta}(\underline{x}_2))|0\rangle = \langle 0|\boldsymbol{\psi}_{a}(\underline{x}_1)\boldsymbol{\psi}_{\beta}(\underline{x}_2)|0\rangle, \tag{5.5.10}$$

which contains terms with $\langle 0|\boldsymbol{b}_{r_1}(\vec{p}_1)\boldsymbol{d}_{r_2}^{\dagger}(\vec{p}_2)|0\rangle$ due to $(3.3.29)$. From $(3.3.27)$ we take

$$\boldsymbol{b}_{r_1}(\vec{p}_1)\boldsymbol{d}_{r_2}^{\dagger}(\vec{p}_2) = -\boldsymbol{d}_{r_2}^{\dagger}(\vec{p}_2)\boldsymbol{b}_{r_1}(\vec{p}_1), \text{ which yields} \tag{5.5.11}$$

$$\langle 0|\boldsymbol{b}_{r_1}(\vec{p}_1)\boldsymbol{d}_{r_2}^{\dagger}(\vec{p}_2)|0\rangle = -\langle 0|\boldsymbol{d}_{r_2}^{\dagger}(\vec{p}_2)\boldsymbol{b}_{r_1}(\vec{p}_1)|0\rangle = 0$$
$$\text{and } \underbrace{\boldsymbol{\psi}_{\alpha}(x_1)\boldsymbol{\psi}}_{}{}_{\beta}(x_2) = 0, \tag{5.5.12}$$

since all other terms vanish.

For $t_1 < t_2$ one gets the same result. Analogously there holds

$$\underbracket{\boldsymbol{\psi}_{\alpha}^{\dagger}(x_1)\boldsymbol{\psi}_{\beta}^{\dagger}}(x_2) = 0 \tag{5.5.13}$$

$$\underbracket{\boldsymbol{\psi}_{\alpha}\boldsymbol{A}}^{(\mu)} = \underbracket{\boldsymbol{\psi}_{\alpha}^{\dagger}\boldsymbol{A}}^{(\mu)} = 0\,. \tag{5.5.14}$$

After these preparations we are able now to write down the Wick expansion of the ***S*** operator (in fact the ***S*** operator of quantum electrodynamics) in second order, $(5.5.7)$. According to $(5.4.38)$ there is one term without a contraction, three terms each involving one and three terms involving two contractions, and one fully contracted term:

$$\boldsymbol{S}^{(2)} = \frac{(\text{-ie})^2}{2!}\int d^4x_1 d^4x_2$$

$$\cdot(:\sum_{\mu,\nu=0}^{0}\frac{g_{\mu\mu}}{\hbar}\frac{g_{\nu\nu}}{\hbar}\underline{\boldsymbol{\psi}}^{\dagger}(\underline{x}_1)\underline{\underline{\alpha}}_{\mu}\underline{\boldsymbol{\psi}}(\underline{x}_1)\underline{\boldsymbol{\psi}}^{\dagger}(\underline{x}_2)\underline{\underline{\alpha}}_{\nu}\underline{\boldsymbol{\psi}}(\underline{x}_2)\boldsymbol{A}^{(\mu)}(\underline{x}_1)\boldsymbol{A}^{(\nu)}(\underline{x}_2): \tag{5.5.15}$$

$$+:\sum_{\mu,\nu=0}^{0}\frac{g_{\mu\mu}}{\hbar}\frac{g_{\nu\nu}}{\hbar}\underline{\boldsymbol{\psi}}^{\dagger}(\underline{x}_1)\underline{\underline{\alpha}}_{\mu}\underbracket{\underline{\boldsymbol{\psi}}(\underline{x}_1)\underline{\boldsymbol{\psi}}^{\dagger}}(\underline{x}_2)\underline{\underline{\alpha}}_{\nu}\underline{\boldsymbol{\psi}}(\underline{x}_2)\boldsymbol{A}^{(\mu)}(\underline{x}_1)\boldsymbol{A}^{(\nu)}(\underline{x}_2): \tag{5.5.15b}$$

$$+:\sum_{\mu,\nu=0}^{0}\frac{g_{\mu\mu}}{\hbar}\frac{g_{\nu\nu}}{\hbar}\underbracket{\underline{\boldsymbol{\psi}}^{\dagger}(\underline{x}_1)\underline{\underline{\alpha}}_{\mu}\underline{\boldsymbol{\psi}}(\underline{x}_1)\underline{\boldsymbol{\psi}}^{\dagger}(\underline{x}_2)\underline{\underline{\alpha}}_{\nu}\underline{\boldsymbol{\psi}}}(\underline{x}_2)\boldsymbol{A}^{(\mu)}(\underline{x}_1)\boldsymbol{A}^{(\nu)}(\underline{x}_2): \tag{5.5.15c}$$

$$+:\sum_{\mu,\nu=0}^{0}\frac{g_{\mu\mu}}{\hbar}\frac{g_{\nu\nu}}{\hbar}\underline{\boldsymbol{\psi}}^{\dagger}(\underline{x}_1)\underline{\underline{\alpha}}_{\mu}\underline{\boldsymbol{\psi}}(\underline{x}_1)\underline{\boldsymbol{\psi}}^{\dagger}(\underline{x}_2)\underline{\underline{\alpha}}_{\nu}\underline{\boldsymbol{\psi}}(\underline{x}_2)\underbracket{\boldsymbol{A}^{(\mu)}(\underline{x}_1)\boldsymbol{A}}^{(\nu)}(\underline{x}_2): \tag{5.5.15d}$$

$$+:\sum_{\mu,\nu=0}^{0}\frac{g_{\mu\mu}}{\hbar}\frac{g_{\nu\nu}}{\hbar}\underline{\boldsymbol{\psi}}^{\dagger}(\underline{x}_1)\underline{\underline{\alpha}}_{\mu}\underbracket{\underline{\boldsymbol{\psi}}(\underline{x}_1)\underline{\boldsymbol{\psi}}^{\dagger}}(\underline{x}_2)\underline{\underline{\alpha}}_{\nu}\underline{\boldsymbol{\psi}}(\underline{x}_2)\underbracket{\boldsymbol{A}^{(\mu)}(\underline{x}_1)\boldsymbol{A}}^{(\nu)}(\underline{x}_2): \tag{5.5.15e}$$

$$+:\sum_{\mu,\nu=0}^{0}\frac{g_{\mu\mu}}{\hbar}\frac{g_{\nu\nu}}{\hbar}\underbracket{\underline{\boldsymbol{\psi}}^{\dagger}(\underline{x}_1)\underline{\underline{\alpha}}_{\mu}\underline{\boldsymbol{\psi}}(\underline{x}_1)\underline{\boldsymbol{\psi}}^{\dagger}(\underline{x}_2)\underline{\underline{\alpha}}_{\nu}\underline{\boldsymbol{\psi}}}(\underline{x}_2)\underbracket{\boldsymbol{A}^{(\mu)}(\underline{x}_1)\boldsymbol{A}}^{(\nu)}(\underline{x}_2): \tag{5.5.15f}$$

$$+:\sum_{\mu,\nu=0}^{0}\frac{g_{\mu\mu}}{\hbar}\frac{g_{\nu\nu}}{\hbar}\underbracket{\underline{\boldsymbol{\psi}}^{\dagger}(\underline{x}_1)\underline{\underline{\alpha}}_{\mu}\underbracket{\underline{\boldsymbol{\psi}}(\underline{x}_1)\underline{\boldsymbol{\psi}}^{\dagger}}(\underline{x}_2)\underline{\underline{\alpha}}_{\nu}\underline{\boldsymbol{\psi}}}(\underline{x}_2)\boldsymbol{A}^{(\mu)}(\underline{x}_1)\boldsymbol{A}^{(\nu)}(\underline{x}_2): \tag{5.5.15g}$$

$$+:\sum_{\mu,\nu=0}^{0}\frac{g_{\mu\mu}}{\hbar}\frac{g_{\nu\nu}}{\hbar}\underbracket{\underline{\boldsymbol{\psi}}^{\dagger}(\underline{x}_1)\underline{\underline{\alpha}}_{\mu}\underbracket{\underline{\boldsymbol{\psi}}(\underline{x}_1)\underline{\boldsymbol{\psi}}^{\dagger}}(\underline{x}_2)\underline{\underline{\alpha}}_{\nu}\underline{\boldsymbol{\psi}}}(\underline{x}_2)\underbracket{\boldsymbol{A}^{(\mu)}(\underline{x}_1)\boldsymbol{A}}^{(\nu)}(\underline{x}_2):)\,. \tag{5.5.15h}$$

Contractions of Fermion spinors stand for the totality of contractions of the involved spinor components.

The expressions $(5.5.15\text{b})$ and $(5.5.15\text{c})$ obtained from the decomposition of the $\boldsymbol{S}$ operator are identical. This is true because using $(5.4.25)$ we obtain from $(5.5.15\text{b})$

$$
\begin{aligned}
&:\underline{\boldsymbol{\psi}}^{\dagger}(\underline{x}_1)\underline{\underline{\alpha}}_{\mu}\underbrace{\underline{\boldsymbol{\psi}}(\underline{x}_1)\underline{\boldsymbol{\psi}}^{\dagger}(\underline{x}_2)}\underline{\underline{\alpha}}_{\nu}\underline{\boldsymbol{\psi}}(\underline{x}_2): \\
&= \varepsilon^2 :\underline{\boldsymbol{\psi}}^{\dagger}(\underline{x}_1)\underline{\underline{\alpha}}_{\mu}\underline{\underline{\alpha}}_{\nu}\underline{\boldsymbol{\psi}}(\underline{x}_2):\underbrace{\underline{\boldsymbol{\psi}}(\underline{x}_1)\underline{\boldsymbol{\psi}}^{\dagger}(\underline{x}_2)}\,.
\end{aligned}
$$

In $(5.5.15\text{c})$ we rename $\underline{x}_1 \rightleftarrows \underline{x}_2$ and $\mu \rightleftarrows \nu$ and exchange the ordering of the field operators

$$
\begin{aligned}
&:\underbrace{\underline{\boldsymbol{\psi}}^{\dagger}(\underline{x}_1)\underline{\underline{\alpha}}_{\mu}\underline{\boldsymbol{\psi}}(\underline{x}_1)\underline{\boldsymbol{\psi}}^{\dagger}(\underline{x}_2)\underline{\underline{\alpha}}_{\nu}\underline{\boldsymbol{\psi}}(\underline{x}_2)}: \\
&:\underbrace{\underline{\boldsymbol{\psi}}^{\dagger}(\underline{x}_2)\underline{\underline{\alpha}}_{\nu}\underline{\boldsymbol{\psi}}(\underline{x}_2)\underline{\boldsymbol{\psi}}^{\dagger}(\underline{x}_1)\underline{\underline{\alpha}}_{\mu}\underline{\boldsymbol{\psi}}(\underline{x}_1)}: \\
&\varepsilon^4 :\underline{\boldsymbol{\psi}}^{\dagger}(\underline{x}_1)\underline{\underline{\alpha}}_{\mu}\underline{\underline{\alpha}}_{\nu}\underline{\boldsymbol{\psi}}(\underline{x}_2):\underbrace{\underline{\boldsymbol{\psi}}(\underline{x}_1)\underline{\boldsymbol{\psi}}^{\dagger}(\underline{x}_2)}\,.
\end{aligned}
$$

In the next sections we will calculate the matrix elements S_{fi} of electro dynamical processes. This type of matrix elements dominates the cross section of the corresponding reactions of particles or quanta (c.f. sections 6.1 and 6.2). Due to $(5.3.5)$, the element of S_{fi} is defined by means of the $\boldsymbol{S}$ operator

$$
S_{fi} = \langle \Phi_f | \boldsymbol{S} | \Phi_i \rangle\,. \tag{5.5.16}
$$

The initial state $|\Phi_i\rangle$ of the system is mostly given by the experimental procedure. One is interested in the frequency of the final state $\langle \Phi_f|$, which can be investigated experimentally. As mentioned in the area of $(5.3.1)$ these two states are defined by their quantum numbers and can be written like this

$$
|\Phi_i\rangle = |\cdots,\underline{p}_m r_m,\cdots,\underline{\overline{p}}_n \overline{r}_n,\cdots,\underline{k}_l\lambda_l,\cdots\rangle\,. \tag{5.5.17}
$$

Here $\underline{p}_m r_m$, $\underline{\overline{p}}_n \overline{r}_n$ and $\underline{k}_l\lambda_l$ designate the momenta and the ordinal numbers of the electrons, positrons and photons. The many-particle states are constructed by applying the creation operators on the vacuum

$$
|\Phi_i\rangle = \cdots\boldsymbol{b}^{\dagger}_{r_m}(\underline{p}_i)\cdots\boldsymbol{d}^{\dagger}_{r_n}(\underline{p}_n)\cdots\boldsymbol{a}^{\dagger}_{\lambda_l}(\underline{k}_l)\cdots|0\rangle\,. \tag{5.5.18}
$$

Normalizing factors can be added on the right hand side in order to generate the correct dimension of S_{fi} (see $(5.6.1)$ and $(5.7.1)$. For the final state analogously holds

$$
\langle \Phi_f| = \langle 0|\cdots,\boldsymbol{a}_{\lambda_l'}(\underline{k}_l'),\cdots,\boldsymbol{d}_{r_n'}(\underline{p}_n'),\cdots,\boldsymbol{b}_{r_m'}(\underline{p}_m'),\cdots\,. \tag{5.5.19}
$$

In the following calculations which are based on $\boldsymbol{S}^{(2)}$, $(5.5.7),(5.5.15)$, in every state only two particles or a particle plus a photon will appear.

5.6 Electron-electron scattering

We study the process of so-called Møller scattering in which two electrons with momenta and ordinal numbers $\underline{p}_1\, r_1$ and $\underline{p}_2\, r_2$ in the initial state are scattered into the final state with $\underline{p}_1'\, r_1'$ and $\underline{p}_2'\, r_2'$. According to $(5.5.16)$, $(5.5.18)$, $(5.5.19)$ and $(5.5.8)$ the $\boldsymbol{S}$ matrix element is given by

$$\begin{aligned} S_{fi} &= \langle 0|\boldsymbol{b}_{r_2'}(\underline{p}_2')\boldsymbol{b}_{r_1'}(\underline{p}_1')\boldsymbol{S}\,\boldsymbol{b}^\dagger_{r_1}(\underline{p}_1)\boldsymbol{b}^\dagger_{r_2}(\underline{p}_2)|0\rangle\mathcal{E}'^4 \\ &\simeq \langle 0|\boldsymbol{b}_{r_2'}(\underline{p}_2')\boldsymbol{b}_{r_1'}(\underline{p}_1')\,\boldsymbol{b}^\dagger_{r_1}(\underline{p}_1)\boldsymbol{b}^\dagger_{r_2}(\underline{p}_2)|0\rangle\mathcal{E}'^4 \\ &+\langle 0|\boldsymbol{b}_{r_2'}(\underline{p}_2')\boldsymbol{b}_{r_1'}(\underline{p}_1')\boldsymbol{S}^{(2)}\,\boldsymbol{b}^\dagger_{r_1}(\underline{p}_1)\boldsymbol{b}^\dagger_{r_2}(\underline{p}_2)|0\rangle\mathcal{E}'^4. \end{aligned} \tag{5.6.1}$$

The operator $\boldsymbol{S}$ and the matrix element S_{fi} are dimensionless. However, due to $(3.3.8)$ the operators $\boldsymbol{b}$ and $\boldsymbol{b}^\dagger$ have the dimension $\left[\text{momentum}^{-3/2}\right]$. Therefore the normalizing factors $\mathcal{E}'$ with dimension $\left[\text{momentum}^{3/2}\right]$ are added. The penultimate term can be left out. We have to consider that outgoing electrons which move in the direction of the incoming ones or oppositely aren't measured, i.e.

$$\underline{p}_1' \neq \underline{p}_1 \quad \text{and} \quad \boldsymbol{b}_{r_1'}(\underline{p}_1')\boldsymbol{b}^\dagger_{r_1}(\underline{p}_1) = -\boldsymbol{b}^\dagger_{r_1}(\underline{p}_1)\boldsymbol{b}_{r_1'}(\underline{p}_1')$$

according to $(3.3.8)$. After further permutations one obtains due to $\underline{p}_2' \neq \underline{p}_1$

$$\begin{aligned} &\langle 0|\boldsymbol{b}_{r_2'}(\underline{p}_2')\boldsymbol{b}_{r_1'}(\underline{p}_1')\boldsymbol{b}^\dagger_{r_1}(\underline{p}_1)\boldsymbol{b}^\dagger_{r_2}(\underline{p}_2)|0\rangle \\ &= -\langle 0|\boldsymbol{b}^\dagger_{r_1}(\underline{p}_1)\boldsymbol{b}_{r_2'}(\underline{p}_2')\boldsymbol{b}^\dagger_{r_2}(\underline{p}_2)\boldsymbol{b}_{r_1'}(\underline{p}_1')|0\rangle = 0. \end{aligned} \tag{5.6.2}$$

Therefore, the last term in $(5.6.1)$ is the second order S matrix element for electron-electron scattering. We claim that the expression $(5.5.15\text{d})$ has to be inserted for $\boldsymbol{S}^{(2)}$ in S_{fi} like this

$$\begin{aligned} S_{fi} &\simeq \frac{(\text{-ie})}{2!}\int d^4x_1 d^4x_2 \\ &\cdot\langle 0|\boldsymbol{b}_{r_2'}(\underline{p}_2')\boldsymbol{b}_{r_1'}(\underline{p}_1')\sum_{\mu,\nu=0}^{3} :\frac{g_{\mu\mu}}{\hbar}\underline{\boldsymbol{\psi}}^\dagger(\underline{x}_1)\underline{\underline{\alpha}}_\mu\underline{\boldsymbol{\psi}}(\underline{x}_1)\frac{g_{\nu\nu}}{\hbar}\underline{\boldsymbol{\psi}}^\dagger(\underline{x}_2)\underline{\underline{\alpha}}_\nu\underline{\boldsymbol{\psi}}(\underline{x}_2): \\ &\cdot\underbrace{\boldsymbol{A}^{(\mu)}(\underline{x}_1)\boldsymbol{A}^{(\nu)}}(\underline{x}_2)\boldsymbol{b}^\dagger_{r_1}(\underline{p}_1)\boldsymbol{b}^\dagger_{r_2}(\underline{p}_2)|0\rangle\mathcal{E}'^4 \end{aligned} \tag{5.6.3}$$

According to $(5.4.1)$ every field operator $\boldsymbol{\psi}$ or $\boldsymbol{\psi}^\dagger$ can be written as a sum of two parts. Due to $(5.4.2)$, in $(5.6.3)$ the part $\boldsymbol{\psi}^-$ contains the operator $\boldsymbol{d}^\dagger_\rho(\vec{q})$ and $\boldsymbol{\psi}^{\dagger+}$ comprises $\boldsymbol{d}_\rho(\vec{q})$, where we have replaced $(\vec{p},r)$ by $(\vec{q},\rho)$. Normal ordering brings all $\boldsymbol{\psi}^-$'s and $\boldsymbol{\psi}^{\dagger-}$'s to the left and all $\boldsymbol{\psi}^+$'s and $\boldsymbol{\psi}^{\dagger+}$'s to the right i.e. the $\boldsymbol{d}^\dagger$'s are moved completely to the left and the $\boldsymbol{d}$'s to the right, for which reason all summands in $(5.6.3)$ vanish which contain $\boldsymbol{d}^\dagger$, $\boldsymbol{d}$ or both. Therefore, in the matrix element $(5.6.3)$ there remain only the partial operators $\boldsymbol{\psi}^{+}$ and $\boldsymbol{\psi}^{\dagger-}$, which carry $\boldsymbol{b}$ or $\boldsymbol{b}^{\dagger}$ operators. We write them down

$$\begin{aligned} S_{fi} \simeq & \frac{(-\mathrm{i}e)^2}{2!}\int d^4x_1 d^4x_2 \langle 0|\boldsymbol{b}_{r_2'}(\underline{p}_2')\boldsymbol{b}_{r_1'}(\underline{p}_1') \\ & \cdot\sum_{\mu,\nu=0}^{3}\frac{g_{\mu\mu}}{\hbar}:\underline{\boldsymbol{\psi}}^{\dagger-}(\underline{x}_1)\underline{\underline{\alpha}}_{\mu}\underline{\boldsymbol{\psi}}^{+}(\underline{x}_1)\frac{g_{\nu\nu}}{\hbar}\underline{\boldsymbol{\psi}}^{\dagger-}(\underline{x}_2)\underline{\underline{\alpha}}_{\nu}\underline{\boldsymbol{\psi}}^{+}(\underline{x}_2): \\ & \cdot\boldsymbol{A}^{(\mu)}(\underline{x}_1)\boldsymbol{A}^{(\nu)}(\underline{x}_2)\boldsymbol{b}_{r_1}^{\dagger}(\underline{p}_1)\boldsymbol{b}_{r_2}^{\dagger}(\underline{p}_2)|0\rangle\mathcal{E}'^4 \end{aligned} \tag{5.6.4}$$

Using the plane wave expansion $(5.4.2)$ and the expression $(5.4.41)$ we arrive at

$$\begin{aligned} S_{fi} \simeq & \frac{(-\mathrm{i}e)^2}{2!\,\hbar^2}\int d^4x_1 d^4x_2 \sum_{\rho_1,\rho_2,\rho_3,\rho_4=1}^{2}\int\frac{d^3q_1}{\mathcal{E}\sqrt{V}}\sqrt{\frac{mc^2}{E(\vec{q}_1)}}\int\frac{d^3q_2}{\mathcal{E}\sqrt{V}}\sqrt{\frac{mc^2}{E(\vec{q}_2)}} \\ & \cdot\int\frac{d^3q_3}{\mathcal{E}\sqrt{V}}\sqrt{\frac{mc^2}{E(\vec{q}_3)}}\int\frac{d^3q_4}{\mathcal{E}\sqrt{V}}\sqrt{\frac{mc^2}{E(\vec{q}_4)}} \\ & \cdot\sum_{\mu,\nu=0}^{3}\underline{w}_{\rho_1}^{\dagger}(\underline{q}_1)e^{-\mathrm{i}\left(\vec{q}_1\vec{x}_1-E(\vec{q}_1)x_1^0/c\right)/\hbar}\underline{\underline{\alpha}}_{\mu}g_{\mu\mu}\underline{w}_{\rho_2}(\underline{q}_2)e^{\mathrm{i}\left(\vec{q}_2\vec{x}_1-E(\vec{q}_2)x_1^0/c\right)/\hbar} \\ & \cdot\underline{w}_{\rho_3}^{\dagger}(\underline{q}_3)e^{-\mathrm{i}\left(\vec{q}_3\vec{x}_2-E(\vec{q}_3)x_2^0/c\right)/\hbar}\underline{\underline{\alpha}}_{\nu}g_{\nu\nu}\underline{w}_{\rho_4}(\underline{q}_4)e^{\mathrm{i}\left(\vec{q}_4\vec{x}_2-E(\vec{q}_4)x_2^0/c\right)/\hbar} \\ & \cdot\mathrm{i}D_F^{\mu\nu}(\underline{x}_1-\underline{x}_2)\mathcal{E}'^4 \\ & \cdot\langle 0|\boldsymbol{b}_{r_2'}(\underline{p}_2')\boldsymbol{b}_{r_1'}(\underline{p}_1'):\boldsymbol{b}_{\rho_1}^{\dagger}(\underline{q}_1)\boldsymbol{b}_{\rho_2}(\underline{q}_2)\boldsymbol{b}_{\rho_3}^{\dagger}(\underline{q}_3)\boldsymbol{b}_{\rho_4}(\underline{q}_4):\boldsymbol{b}_{r_1}^{\dagger}(\underline{p}_1)\boldsymbol{b}_{r_2}^{\dagger}(\underline{p}_2)|0\rangle. \end{aligned} \tag{5.6.5}$$

we resolve the normal ordering in the vacuum expectation value. The product $:\boldsymbol{b}_{\rho_1}^{\dagger}(\underline{q}_1)\boldsymbol{b}_{\rho_2}(\underline{q}_2)\boldsymbol{b}_{\rho_3}^{\dagger}(\underline{q}_3)\boldsymbol{b}_{\rho_4}(\underline{q}_4):$ equals to $-\boldsymbol{b}_{\rho_1}^{\dagger}(\underline{q}_1)\boldsymbol{b}_{\rho_3}^{\dagger}(\underline{q}_3)\boldsymbol{b}_{\rho_2}(\underline{q}_2)\boldsymbol{b}_{\rho_4}(\underline{q}_4)$, which we insert in the vacuum expectation value, and we bring all $\boldsymbol{b}^{\dagger}$'s to the left and all $\boldsymbol{b}$'s to the right using $(3.3.8)$ like this

$$\begin{aligned} & -\langle 0|\boldsymbol{b}_{r_2'}(\underline{p}_2')\boldsymbol{b}_{r_1'}(\underline{p}_1')\boldsymbol{b}_{\rho_1}^{\dagger}(\underline{q}_1)\boldsymbol{b}_{\rho_3}^{\dagger}(\underline{q}_3)\boldsymbol{b}_{\rho_2}(\underline{q}_2)\boldsymbol{b}_{\rho_4}(\underline{q}_4)\boldsymbol{b}_{r_1}^{\dagger}(\underline{p}_1)\boldsymbol{b}_{r_2}^{\dagger}(\underline{p}_2)|0\rangle \\ & = -\langle 0|\boldsymbol{b}_{r_2'}(\underline{p}_2')\left(-\boldsymbol{b}_{\rho_1}^{\dagger}(\underline{q}_1)\boldsymbol{b}_{r_1'}(\underline{p}_1')+\delta_{r_1'\rho_1}\delta^3(\vec{p}_1'-\vec{q}_1)\right)\boldsymbol{b}_{\rho_3}^{\dagger}(\underline{q}_3)\cdots \\ & = -\langle 0|\left(\left(\boldsymbol{b}_{\rho_1}^{\dagger}(\underline{q}_1)\boldsymbol{b}_{r_2'}(\underline{p}_2')-\delta_{r_2'\rho_1}\delta^3(\vec{p}_2'-\vec{q}_1)\right)\boldsymbol{b}_{r_1'}(\underline{p}_1')\boldsymbol{b}_{\rho_3}^{\dagger}(\underline{q}_3)\right. \\ & \qquad \left.+\delta_{r_1'\rho_1}\delta^3(\vec{p}_1'-\vec{q}_1)\boldsymbol{b}_{r_2'}(\underline{p}_2')\boldsymbol{b}_{\rho_3}^{\dagger}(\underline{q}_3)\right)\cdots \\ & = -\langle 0|\left(-\delta_{r_2'\rho_1}\delta^3(\vec{p}_2'-\vec{q}_1)\left(-\boldsymbol{b}_{\rho_3}^{\dagger}(\underline{q}_3)\boldsymbol{b}_{r_1'}(\underline{p}_1')+\delta_{r_1'\rho_3}\delta^3(\vec{p}_1'-\vec{q}_3)\right)\right. \\ & \qquad \left.+\delta_{r_1'\rho_1}\delta^3(\vec{p}_1'-\vec{q}_1)\left(-\boldsymbol{b}_{\rho_3}^{\dagger}(\underline{q}_3)\boldsymbol{b}_{r_2'}(\underline{p}_2')+\delta_{r_2'\rho_3}\delta^3(\vec{p}_2'-\vec{q}_3)\right)\right)\cdots \\ & = \left(\delta_{r_1'\rho_3}\delta^3(\vec{p}_1'-\vec{q}_3)\cdot\delta_{r_2'\rho_1}\delta^3(\vec{p}_2'-\vec{q}_1)-\delta_{r_1'\rho_1}\delta^3(\vec{p}_1'-\vec{q}_1)\delta_{r_2'\rho_3}\delta^3(\vec{p}_2'-\vec{q}_3)\right) \\ & \cdot\left(\delta_{r_1\rho_4}\delta^3(\vec{p}_1-\vec{q}_4)\delta_{r_2\rho_2}\delta^3(\vec{p}_2-\vec{q}_2)-\delta_{r_1\rho_2}\delta^3(\vec{p}_1-\vec{q}_2)\delta_{r_2\rho_4}\delta^3(\vec{p}_2-\vec{q}_4)\right). \end{aligned} \tag{5.6.6}$$

Using $(3.4.8)$ and $(5.6.6)$ we write the $\boldsymbol{S}$ matrix element like this

$$S_{fi} \simeq \frac{(-\mathrm{i}e)^2}{2!\,\hbar^2 V^2}\int d^4x_1 d^4x_2 \frac{m^2c^4}{\sqrt{E(\vec{p}_1)E(\vec{p}_2)E(\vec{p}_1')E(\vec{p}_2')}}\left(\frac{\mathcal{E}'}{\mathcal{E}}\right)^4$$
$$\cdot\sum_{\mu,\nu=0}^{3}\mathrm{i}D_F^{\mu\nu}(\underline{x}_1-\underline{x}_2)$$
$$\cdot[e^{\mathrm{i}(\underline{p}_2'-\underline{p}_2)\underline{x}_1/\hbar}e^{\mathrm{i}(\underline{p}_1'-\underline{p}_1)\underline{x}_2/\hbar}\underline{w}_{r_2'}^{\dagger}(\underline{p}_2')\underline{\underline{\alpha}}_{\mu}g_{\mu\mu}\underline{w}_{r_2}(\underline{p}_2)\underline{w}_{r_1'}^{\dagger}(\underline{p}_1')\underline{\underline{\alpha}}_{\nu}g_{\nu\nu}\underline{w}_{r_1}(\underline{p}_1) \qquad (5.6.7)$$
$$-e^{\mathrm{i}(\underline{p}_2'-\underline{p}_1)\underline{x}_1/\hbar}e^{\mathrm{i}(\underline{p}_1'-\underline{p}_2)\underline{x}_2/\hbar}\underline{w}_{r_2'}^{\dagger}(\underline{p}_2')\underline{\underline{\alpha}}_{\mu}g_{\mu\mu}\underline{w}_{r_1}(\underline{p}_1)\underline{w}_{r_1'}^{\dagger}(\underline{p}_1')\underline{\underline{\alpha}}_{\nu}g_{\nu\nu}\underline{w}_{r_2}(\underline{p}_2)$$
$$-e^{\mathrm{i}(\underline{p}_1'-\underline{p}_2)\underline{x}_1/\hbar}e^{\mathrm{i}(\underline{p}_2'-\underline{p}_1)\underline{x}_2/\hbar}\underline{w}_{r_1'}^{\dagger}(\underline{p}_1')\underline{\underline{\alpha}}_{\mu}g_{\mu\mu}\underline{w}_{r_2}(\underline{p}_2)\underline{w}_{r_2'}^{\dagger}(\underline{p}_2')\underline{\underline{\alpha}}_{\nu}g_{\nu\nu}\underline{w}_{r_1}(\underline{p}_1)$$
$$+e^{\mathrm{i}(\underline{p}_1'-\underline{p}_1)\underline{x}_1/\hbar}e^{\mathrm{i}(\underline{p}_2'-\underline{p}_2)\underline{x}_2/\hbar}\underline{w}_{r_1'}^{\dagger}(\underline{p}_1')\underline{\underline{\alpha}}_{\mu}g_{\mu\mu}\underline{w}_{r_1}(\underline{p}_1)\underline{w}_{r_2'}^{\dagger}(\underline{p}_2')\underline{\underline{\alpha}}_{\nu}g_{\nu\nu}\underline{w}_{r_2}(\underline{p}_2)]$$

Equation $(4.7.9)$ reveals that in $D_F^{\mu\nu}(\underline{x}_1-\underline{x}_2)$ the indices μ and ν can be interchanged and $\underline{x}_1$ can be interchanged with $\underline{x}_2$. Therefore, the third term in $(5.6.7)$ is identical to the second and the fourth term is identical to the first and we can write

$$S_{fi} \simeq \frac{(-\mathrm{i}e)^2}{\hbar^2 V^2}\int d^4x_1 d^4x_2 \frac{m^2c^4}{\sqrt{E(\vec{p}_1)E(\vec{p}_2)E(\vec{p}_1')E(\vec{p}_2')}}\left(\frac{\mathcal{E}'}{\mathcal{E}}\right)^4$$
$$\cdot\sum_{\mu,\nu=0}^{3}\mathrm{i}D_F^{\mu\nu}(\underline{x}_1-\underline{x}_2) \qquad (5.6.8)$$
$$\cdot[e^{\mathrm{i}(\underline{p}_2'-\underline{p}_2)\underline{x}_2/\hbar}e^{\mathrm{i}(\underline{p}_1'-\underline{p}_1)\underline{x}_1/\hbar}\underline{w}_{r_2'}^{\dagger}(\underline{p}_2')\underline{\underline{\alpha}}_{\mu}g_{\mu\mu}\underline{w}_{r_2}(\underline{p}_2)\underline{w}_{r_1'}^{\dagger}(\underline{p}_1')\underline{\underline{\alpha}}_{\nu}g_{\nu\nu}\underline{w}_{r_1}(\underline{p}_1)$$
$$-e^{\mathrm{i}(\underline{p}_2'-\underline{p}_1)\underline{x}_2/\hbar}e^{\mathrm{i}(\underline{p}_1'-\underline{p}_2)\underline{x}_1/\hbar}\underline{w}_{r_2'}^{\dagger}(\underline{p}_2')\underline{\underline{\alpha}}_{\mu}g_{\mu\mu}\underline{w}_{r_1}(\underline{p}_1)\underline{w}_{r_1'}^{\dagger}(\underline{p}_1')\underline{\underline{\alpha}}_{\nu}g_{\nu\nu}\underline{w}_{r_2}(\underline{p}_2)].$$

The expression $(5.6.8)$ is a difference of two terms. The first one is named direct term and the second is the exchange term.

With the Fourier transform $(4.7.9\,/\,10)$

$$D_F^{\mu\nu}(\underline{x}_1-\underline{x}_2)=\int\frac{d^4q}{(2\pi)^4}e^{-\mathrm{i}\underline{q}(\underline{x}_1-\underline{x}_2)}D_F^{\mu\nu}(\underline{q}) \quad \text{with}$$
$$D_F^{\mu\nu}(\underline{q})=\frac{-g_{\mu\nu}}{\underline{q}^2+\mathrm{i}\varepsilon}\frac{\hbar}{\varepsilon_0 c} \qquad (5.6.9)$$

the part of the direct term which contains $\underline{x}_1$ and $\underline{x}_2$ becomes

$$J=\int d^4x_1 d^4x_2 e^{\mathrm{i}(\underline{p}_2'-\underline{p}_2)\underline{x}_2/\hbar}e^{\mathrm{i}(\underline{p}_1'-\underline{p}_1)\underline{x}_1/\hbar}\int\frac{d^4q}{(2\pi)^4}e^{-\mathrm{i}\underline{q}(\underline{x}_1-\underline{x}_2)}D_F^{\mu\nu}(\underline{q})$$
$$=\int d^4x_1\int\frac{d^4q}{(2\pi)^4}e^{\mathrm{i}(\underline{p}_1'-\underline{p}_1-\underline{q}\hbar)\underline{x}_1/\hbar}D_F^{\mu\nu}(\underline{q})\int d^4x_2 e^{(\underline{p}_2'-\underline{p}_2+\underline{q}\hbar)\underline{x}_2/\hbar}.$$

With the substitution $y_2 = x_2 / \hbar$ and the four dimensional form of $(3.2.25)$ the relation $\int d^4 x_2 e^{\mathrm{i}(\underline{p}_2' - \underline{p}_2 + \underline{k}\hbar)\underline{x}_2/\hbar} = \hbar^4 (2\pi)^4 \delta^4 (\underline{p}_2' - \underline{p}_2 + \underline{k}\hbar)$ holds and we obtain

$$J = \int d^4 x_1 \int \frac{d^4 q}{(2\pi)^4} e^{\mathrm{i}(\underline{p}_1' - \underline{p}_1 - \underline{q}\hbar)x_1/\hbar} D_F^{\mu\nu}(\underline{q}) \hbar^4 (2\pi)^4 \delta^4 (\underline{p}_2' - \underline{p}_2 + \underline{q}\hbar).$$

With the help of $(6.1.33)$ we get

$$\begin{aligned} J &= \int d^4 x_1 e^{\mathrm{i}(\underline{p}_1' - \underline{p}_1 + \underline{p}_2' - \underline{p}_2)\underline{x}_1/\hbar} D_F^{\mu\nu}\left(\frac{\underline{p}_2 - \underline{p}_2'}{\hbar}\right) \\ &= (2\pi)^4 \hbar^4 \delta^4 (\underline{p}_1' - \underline{p}_1 + \underline{p}_2' - \underline{p}_2) D_F^{\mu\nu}\left(\frac{\underline{p}_1 - \underline{p}_1'}{\hbar}\right) \end{aligned} \tag{5.6.10}$$

and similarly for the exchange term

$$\begin{aligned} &\int d^4 x_1 d^4 x_2 e^{\mathrm{i}(\underline{p}_2' - \underline{p}_1)\underline{x}_2/\hbar} e^{\mathrm{i}(\underline{p}_1' - \underline{p}_2)\underline{x}_1/\hbar} \int \frac{d^4 k}{(2\pi)^4} e^{-\mathrm{i}\underline{k}(\underline{x}_1 - \underline{x}_2)} D_F^{\mu\nu}(\underline{k}) \\ &= (2\pi)^4 \hbar^4 \delta^4 (\underline{p}_1' - \underline{p}_2 + \underline{p}_2' - \underline{p}_1) D_F^{\mu\nu}\left(\frac{\underline{p}_1' - \underline{p}_2}{\hbar}\right). \end{aligned} \tag{5.6.11}$$

The δ functions in $(5.6.10/11)$ reveal

$$\underline{p}_1 + \underline{p}_2 = \underline{p}_1' + \underline{p}_2', \tag{5.6.12}$$

$$\begin{aligned} &\text{which means} \quad E(\vec{p}_1) + E(\vec{p}_2) = E(\vec{p}_1') + E(\vec{p}_2') \\ &\text{and} \qquad \vec{\mathrm{p}}_1 + \vec{\mathrm{p}}_2 = \vec{\mathrm{p}}_1' + \vec{\mathrm{p}}_2' \quad , \end{aligned} \tag{5.6.13}$$

i.e. the energy-momentum conservation of the scattering process is satisfied. The S matrix element finally reads

$$\begin{aligned} S_{fi} \simeq &\frac{\mathrm{i}(-\mathrm{i}e)^2}{\hbar^2} \frac{(2\pi)^4 \hbar^4}{V^2} \frac{m^2 c^4}{\sqrt{E(\vec{p}_1)E(\vec{p}_2)E(\vec{p}_1')E(\vec{p}_2')}} \\ &\cdot \left(\frac{\mathcal{E}'}{\mathcal{E}}\right)^4 \delta^4 (\underline{p}_1' + \underline{p}_2' - \underline{p}_1 - \underline{p}_2) \cdot \\ \sum_{\mu\nu=0}^{3} &[\underline{w}_{r_2'}^{\dagger}(\underline{p}_2') \underline{\underline{\alpha}}_{\mu} g_{\mu\mu} \underline{w}_{r_2}(\underline{p}_2) D_F^{\mu\nu}\left(\frac{\underline{p}_1' - \underline{p}_1}{\hbar}\right) \underline{w}_{r_1'}^{\dagger}(\underline{p}_1') \underline{\underline{\alpha}}_{\nu} g_{\nu\nu} \underline{w}_{r_1}(\underline{p}_1) \\ &- \underline{w}_{r_2'}^{\dagger}(\underline{p}_2') \underline{\underline{\alpha}}_{\mu} g_{\mu\mu} \underline{w}_{r_1}(\underline{p}_1) D_F^{\mu\nu}\left(\frac{\underline{p}_1' - \underline{p}_2}{\hbar}\right) \underline{w}_{r_1'}^{\dagger}(\underline{p}_1') \underline{\underline{\alpha}}_{\nu} g_{\nu\nu} \underline{w}_{r_2}(\underline{p}_2)] \end{aligned} \tag{5.6.14}$$

The result $(5.6.14)$ has also been obtained in quantum electrodynamics by using a heuristic propagator formalism. However, field quantization provides a solid theoretical frame work for this and similar results.

For the construction of $(5.6.3)$ we have chosen the S matrix element $(5.5.15\text{d})$ and the result was reduced to $(5.6.5)$ where in the normal ordering domain there is only the same type of creation- and annihilation operators as outside the "normal ordering". Similar arguments which lead to $(5.6.5)$ reveal that this is true for all the reactions which are treated with a S matrix element $(5.5.15)$. We will apply this rule in the next section.

5.7 Compton scattering

The initial state for Compton scattering contains an electron $\underline{p},r$ and a photon $\underline{k},\lambda$, whereas the same particle and quantum in the final state are characterized by $\underline{p}',r'$ and $\underline{k}',\lambda'$. In analogy to $(5.6.1)$ the scattering matrix reads

$$\begin{aligned} S_{fi} &= \langle 0|\boldsymbol{b}_{r'}(\underline{p}')\boldsymbol{a}_{\lambda'}(\underline{k}')\boldsymbol{S}\,\boldsymbol{a}_{\lambda}^{\dagger}(\underline{k})\boldsymbol{b}_{r}^{\dagger}(\underline{p})|0\rangle \mathcal{A}'^2\mathcal{E}'^2 \\ &\simeq \langle 0|\boldsymbol{b}_{r'}(\underline{p}')\boldsymbol{a}_{\lambda'}(\underline{k}')\boldsymbol{S}^{(2)}\,\boldsymbol{a}_{\lambda}^{\dagger}(\underline{k})\boldsymbol{b}_{r}^{\dagger}(\underline{p})|0\rangle \mathcal{A}'^2\mathcal{E}'^2, \end{aligned} \tag{5.7.1}$$

where the unity operator in $\boldsymbol{S}$ could be omitted with arguments similar to $(5.6.1/2)$. In analogy to $(5.6.1)$ a normalizing factor $\mathcal{A}'^2\mathcal{E}'^2$ is added in order to eliminate the dimensions of the $\boldsymbol{a}$'s and $\boldsymbol{b}$'s. The dimension of $\mathcal{A}'$ and $\mathcal{E}'$ is $\left[\text{momentum}^{3/2}\right]$. In order to fulfil the rule given at the end of the preceding section we choose $\boldsymbol{S}^{(2)}$ from $(5.5.15\text{b})$, which is identical to $(5.5.15\text{c})$ as shown above. Therefore a factor 2 results (see also $(3.4.1)$ and consider $\underline{\underline{\beta}}^2 = \underline{\underline{1}}$)

$$\begin{aligned} S_{fi} &\simeq 2\frac{(-ie)^2}{2!}\mathcal{A}'^2\mathcal{E}'^2\sum_{\mu,\nu=0}^{3}\frac{1}{\hbar^2}\int d^4x_1 d^4x_2 \\ &\cdot\langle 0|\boldsymbol{b}_{r'}(\underline{p}')\boldsymbol{a}_{\lambda'}(\underline{k}'):\underline{\boldsymbol{\psi}}^{\dagger}(\underline{x}_1)\underline{\underline{\alpha}}_{\mu}g_{\mu\mu}\underline{\boldsymbol{\psi}}(\underline{x}_1)\cdot\underline{\overline{\boldsymbol{\psi}}}(\underline{x}_2) \\ &\cdot\underline{\underline{\beta}}\underline{\underline{\alpha}}_{\nu}g_{\nu\nu}\underline{\boldsymbol{\psi}}(\underline{x}_2)A^{(\mu)}(\underline{x}_1)A^{(\nu)}(\underline{x}_2):\boldsymbol{a}_{\lambda}^{\dagger}(\underline{k})\boldsymbol{b}_{r}^{\dagger}(\underline{p})|0\rangle. \end{aligned} \tag{5.7.2}$$

With similar arguments as for $(5.6.4)$ from $\underline{\boldsymbol{\psi}}^{\dagger}(\underline{x}_1)$ we use only $\underline{\boldsymbol{\psi}}^{\dagger-}(\underline{x}_1)$, and from $\underline{\boldsymbol{\psi}}(\underline{x}_2)$ only $\underline{\boldsymbol{\psi}}^{+}(\underline{x}_2)$ is kept. In the product $:\boldsymbol{A}^{(\mu)}(\underline{x}_1)\boldsymbol{A}^{(\nu)}(x_2):$ the term $\boldsymbol{A}^{(\mu)+}(\underline{x}_1)\boldsymbol{A}^{(\nu)+}(\underline{x}_2)$ does not contribute to the result because of

$$\begin{aligned} &\langle 0|\boldsymbol{a}(\underline{k}')\boldsymbol{a}(\underline{k}_1)\boldsymbol{a}(\underline{k}_2)\boldsymbol{a}^{\dagger}(\underline{k})|0\rangle = \langle 0|\boldsymbol{a}(\underline{k}_1)\boldsymbol{a}(\underline{k}_2)\boldsymbol{a}(\underline{k}')\boldsymbol{a}^{\dagger}(\underline{k})|0\rangle \\ &= \langle 0|\boldsymbol{a}(\underline{k}_1)\boldsymbol{a}(\underline{k}_2)\left(\boldsymbol{a}^{\dagger}(\underline{k})\boldsymbol{a}(\underline{k}') + \delta^3\left(\vec{k}'-\vec{k}\right)\right)|0\rangle = 0. \end{aligned} \tag{5.7.3}$$

Again, $\vec{k}' = \vec{k}$ is excluded because such quanta $\vec{k}',\lambda'$ aren't measured. An analogous relation is true for $\boldsymbol{A}^{(\mu)-}(\underline{x}_1)\boldsymbol{A}^{(\nu)-}(\underline{x}_2)$. Therefore, only mixed $\boldsymbol{A}^{\pm}$-terms appear in S_{fi} like this

$$S_{fi} \simeq (\mathrm{ie})^2 \sum_{\mu,\nu=0}^{3} \frac{1}{\hbar^2} \int d^4x_1 d^4x_2 \mathcal{A}'^2 \mathcal{E}'^2$$
$$\cdot \langle 0 | \boldsymbol{b}_{r'}(\underline{p}') \boldsymbol{a}_{\lambda'}(\underline{k}') : \underline{\boldsymbol{\psi}}^{\dagger -}(\underline{x}_1) \underline{\underline{\alpha}}_\mu g_{\mu\mu} \underline{\boldsymbol{\psi}}^{+}(\underline{x}_1) \underline{\overline{\boldsymbol{\psi}}}^{-}(\underline{x}_2) \underline{\underline{\beta}} \underline{\underline{\alpha}}_\nu g_{\nu\nu} \underline{\boldsymbol{\psi}}^{+}(\underline{x}_2): \tag{5.7.4}$$
$$\cdot : \left(\boldsymbol{A}^{(\mu)-}(\underline{x}_1) \boldsymbol{A}^{(\nu)+}(\underline{x}_2) + \boldsymbol{A}^{(\mu)+}(\underline{x}_1) \boldsymbol{A}^{(\nu)-}(\underline{x}_2) \right) : \boldsymbol{a}^\dagger_\lambda(\underline{k}) \boldsymbol{b}^\dagger_r(\underline{p}) | 0 \rangle.$$

Using $(3.4.8), (5.4.2), (5.4.3)$ and $(5.4.39)$ we obtain

$$S_{fi} \simeq (-\mathrm{ie})^2 \int d^4x_1 d^4x_2 \sum_{\mu,\nu=0}^{3} \sum_{\rho_1,\rho_2=1}^{2} \sum_{\lambda_1,\lambda_2=0}^{3} \frac{\mathcal{A}'^2 \mathcal{E}'^2}{\hbar^2} \frac{1}{\varepsilon_0 \hbar}$$
$$\cdot \int \frac{d^3q_1}{\mathcal{E}\sqrt{V}} \sqrt{\frac{mc^2}{E(\vec{q}_1)}} \int \frac{d^3q_2}{\mathcal{E}\sqrt{V}} \sqrt{\frac{mc^2}{E(\vec{q}_2)}} \int \frac{d^3k_1}{\sqrt{(2\pi)^3 2\omega_1}} \int \frac{d^3k_2}{\sqrt{(2\pi)^3 2\omega_2}}$$
$$\underline{w}^\dagger_{\rho_1}(\vec{q}_1) e^{\mathrm{i}\underline{q}_1\underline{x}_1/\hbar} \underline{\underline{\alpha}}_\mu g_{\mu\mu} \mathrm{i}\underline{\underline{S}}_F(\underline{x}_1 - \underline{x}_2) \underline{\underline{\beta}}\, \underline{\underline{\alpha}}_\nu g_{\nu\nu} \underline{w}_{\rho_2}(\vec{q}_2) e^{-\mathrm{i}\underline{q}_2\underline{x}_2/\hbar}$$
$$[\varepsilon_{\mu,\lambda_1}(\underline{k}_1) e^{\mathrm{i}\underline{k}_1\underline{x}_1/\hbar} \varepsilon_{\nu,\lambda_2}(\underline{k}_2) e^{-\mathrm{i}\underline{k}_2\underline{x}_2/\hbar} \tag{5.7.5}$$
$$\cdot \langle 0 | \boldsymbol{b}_{r'}(\underline{p}') \boldsymbol{a}_{\lambda'}(\underline{k}') : \boldsymbol{b}^\dagger_{\rho_1}(\underline{q}_1) \boldsymbol{b}_{\rho_2}(\underline{q}_2) \boldsymbol{a}^\dagger_{\lambda_1}(\underline{k}_1) \boldsymbol{a}_{\lambda_2}(\underline{k}_2) : \boldsymbol{a}^\dagger_\lambda(\underline{k}) \boldsymbol{b}^\dagger_r(\underline{p}) | 0 \rangle$$
$$+ \varepsilon_{\mu,\lambda_1}(\underline{k}_1) e^{-\mathrm{i}\underline{k}_1\underline{x}_1/\hbar} \varepsilon_{\nu,\lambda_2}(\underline{k}_2) e^{\mathrm{i}\underline{k}_2\underline{x}_2/\hbar}$$
$$\cdot \langle 0 | \boldsymbol{b}_{r'}(\underline{p}') \boldsymbol{a}_{\lambda'}(\underline{k}') : \boldsymbol{b}^\dagger_{\rho_1}(\underline{q}_1) \boldsymbol{b}_{\rho_2}(\underline{q}_2) \boldsymbol{a}_{\lambda_1}(\underline{k}_1) \boldsymbol{a}^\dagger{}_{\lambda_2}(\underline{k}_2) : \boldsymbol{a}^\dagger_\lambda(\underline{k}) \boldsymbol{b}^\dagger_r(\underline{p}) | 0 \rangle.$$

As in $(5.4.2)$ and $(5.6.5)$ we have gone to the box normalization of electrons i.e. we have replaced $\frac{1}{\sqrt{(2\pi\hbar)^3}}$ by $\frac{1}{\mathcal{E}\sqrt{V}}$ (c.f. $(3.3.30)$). The first vacuum expectation value in $(5.7.5)$ becomes in analogy to $(5.6.6)$

$$\langle 0 | \boldsymbol{b}_{r'}(p') \boldsymbol{a}_{\lambda'}(\underline{k}') : \boldsymbol{b}^\dagger_{\rho_1}(\underline{q}_1) \boldsymbol{b}_{\rho_2}(\underline{q}_2) \boldsymbol{a}^\dagger_{\lambda_1}(\underline{k}_1) \boldsymbol{a}_{\lambda_2}(\underline{k}_2) : \boldsymbol{a}^\dagger_\lambda(\underline{k}) \boldsymbol{b}^\dagger_r(\underline{p}) | 0 \rangle$$
$$= \langle 0 | \boldsymbol{b}_{r'}(p') \boldsymbol{b}^\dagger_{\rho_1}(\underline{q}_1) \boldsymbol{b}_{\rho_2}(\underline{q}_2) \boldsymbol{b}^\dagger_r(\underline{p}) \boldsymbol{a}_{\lambda'}(\underline{k}') \boldsymbol{a}^\dagger_{\lambda_1}(\underline{k}_1) \boldsymbol{a}_{\lambda_2}(\underline{k}_2) \boldsymbol{a}^\dagger_\lambda(\underline{k}) | 0 \rangle \tag{5.7.6}$$
$$= \delta_{\rho_1 r'} \delta^3(\vec{q}_1 - \vec{p}') \delta_{\rho_2 r} \delta^3(\vec{q}_2 - \vec{p}) \delta_{\lambda'\lambda_1} \delta^3(\vec{k}' - \vec{k}_1) \delta_{\lambda\lambda_2} \delta(\vec{k} - \vec{k}_2).$$

The second one reads

$$\delta_{\rho_1 r'} \delta^3(\vec{q}_1 - p') \delta_{\rho_2 r} \delta^3(\vec{q}_2 - \vec{p}) \delta_{\lambda'\lambda_2} \delta^3(\vec{k}' - \vec{k}_2) \delta_{\lambda\lambda_1} \delta(\vec{k} - \vec{k}_1). \tag{5.7.7}$$

We insert $(5.7.6)$ and $(5.7.7)$ in $(5.7.5)$, which yields

$$S_{fi} \simeq \frac{(-\mathrm{ie})^2}{\mathcal{E}^2 V (2\pi)^3} \frac{mc^2}{\sqrt{E(\vec{p}) E(\vec{p}') 2\omega_k 2\omega_{k'}}} \int d^4x_1 d^4x_2 \frac{\mathcal{A}'^2 \mathcal{E}'^2}{\hbar^2 \varepsilon_0 \hbar}$$
$$\cdot \sum_{\mu,\nu=0}^{3} \underline{w}^\dagger_{r'}(\underline{p}') \underline{\underline{\alpha}}_\mu g_{\mu\mu} \mathrm{i}\underline{\underline{S}}_F(\underline{x}_1 - \underline{x}_2) \underline{\underline{\beta}} \underline{\underline{\alpha}}_\nu g_{\nu\nu} \underline{w}_r(\underline{p}) e^{\mathrm{i}\underline{p}'\underline{x}_1/\hbar} e^{-\mathrm{i}\underline{p}\underline{x}_2/\hbar} \tag{5.7.8}$$
$$[\varepsilon_{\mu,\lambda'}(\underline{k}') \varepsilon_{\nu,\lambda}(\underline{k}) e^{\mathrm{i}\underline{k}'\underline{x}_1/\hbar} e^{-\mathrm{i}\underline{k}\underline{x}_2/\hbar} + \varepsilon_{\mu,\lambda}(\underline{k}) \varepsilon_{\nu,\lambda'}(\underline{k}') e^{-\mathrm{i}\underline{k}\underline{x}_1/\hbar} e^{\mathrm{i}\underline{k}'\underline{x}_2/\hbar}].$$

From now on in analogy to $(5.7.5)$ (c.f. $(3.3.30)$) we go to the box normalization for photons, i.e. in S_{fi}, $(5.7.8)$, we replace $\frac{1}{(2\pi)^{3/2}}$ by $\frac{\hbar^{3/2}}{\mathcal{A}V^{1/2}}$. The factor $\mathcal{A}$ has the dimension $\left[\text{momentum}^{3/2}\right]$ in the same way as $\mathcal{E}$.

The quantity $\underline{\underline{S}}_F(\underline{x}_1 - \underline{x}_2)$ is a 4×4-matrix and due to $(3.4.38)$ and $(5.4.40)$ it represents a Fourier transform of the form

$$\begin{aligned}
&\underline{\underline{S}}_F(\underline{x}_1-\underline{x}_2)=\int\frac{d^4q}{(2\pi)^4}e^{-\mathrm{i}\underline{q}(\underline{x}_1-\underline{x}_2)}\underline{\underline{S}}_F(\underline{q})\\
&\text{with }\underline{\underline{S}}_F(\underline{q})=\frac{\underline{\underline{\not{q}}}+\underline{\underline{1}}mc/\hbar}{\underline{q}^2-\left(mc^2/\hbar\right)^2+\mathrm{i}\varepsilon}.
\end{aligned}\tag{5.7.9}$$

Similarly to $(5.6.14)$ we obtain

$$\begin{aligned}
S_{fi}\simeq{}&\mathrm{i}\frac{(-\mathrm{i}e)^2}{V^2}\frac{mc^2}{\sqrt{E(\vec{p})E(\vec{p}\,')2\omega_k2\omega_{k'}}}\\
&\cdot\hbar^4(2\pi)^4\delta^4(\underline{p}'+\underline{k}'-\underline{p}-\underline{k})\frac{1}{\hbar^2}\hbar^3\frac{\mathcal{A}'^2\mathcal{E}'^2}{\mathcal{A}^2\mathcal{E}^2}\frac{1}{\varepsilon_0\hbar}\\
&\cdot\sum_{\mu,\nu=0}^{3}[\underline{w}_{r'}^{\dagger}(\underline{p}')\underline{\underline{\alpha}}_{\mu}g_{\mu\mu}\underline{\underline{S}}_F\left(\frac{\underline{p}+\underline{k}}{\hbar}\right)\underline{\underline{\beta}}\underline{\underline{\alpha}}_{\nu}g_{\nu\nu}\underline{w}_r(\underline{p})\varepsilon_{\mu,\lambda'}(\underline{k}')\varepsilon_{\nu,\lambda}(\underline{k})\\
&\quad+\underline{w}_{r'}^{\dagger}(\underline{p}')\underline{\underline{\alpha}}_{\mu}g_{\mu\mu}\underline{\underline{S}}_F\left(\frac{\underline{p}-\underline{k}'}{\hbar}\right)\underline{\underline{\beta}}\underline{\underline{\alpha}}_{\nu}g_{\nu\nu}\underline{w}_r(\underline{p})\varepsilon_{\mu,\lambda}(\underline{k})\varepsilon_{\nu,\lambda'}(\underline{k}')].
\end{aligned}\tag{5.7.10}$$

the S matrix element is the main ingredient of the scattering cross section, which we will deal with in the next chapter.

6 Scattering cross sections

Cross sections are important quantities in order to verify the quantum field theory (or the quantum electrodynamic theory).
The differential scattering cross section states how many particles or quantas are scattered per target particle per unit solid angle in a certain direction per time unit owing to a given current of incident particles (quantas). It has the dimension of an area.

6.1 The scattering cross section of electrons

In order to describe the number of states the box normalization of wave functions is best suitable. In this model we consider only wave functions in a – relatively large – box. We choose a cubical box of volume $V = L^3$. We demand that – for instance – the x-dependence of a wave function is identical at $x = 0$ and at $x = L$. Thus, for the function $e^{ip_x x/\hbar}$ we demand

$$1 = e^0 = e^{ip_x L/\hbar} \tag{6.1.1}$$

$$\begin{aligned} \text{or} \quad & p_x L/\hbar = n_x 2\pi, \qquad n_x \text{ integer} \\ \text{and } & p_y L/\hbar = n_y 2\pi \\ & p_z L/\hbar = n_z 2\pi. \end{aligned} \tag{6.1.2}$$

The whole number of states for $\vec{p}$ within L^3 is

$$N = n_x n_y n_z = \left(\frac{L}{2\pi\hbar}\right)^3 p_x p_y p_z .$$

(The often given interpretation of this expression as a result of standing waves can be wrong by a factor 2^3). The momentum range $dp_x dp_y dp_z$ comprises

$$dN = \left(\frac{L}{2\pi\hbar}\right)^3 dp_x dp_y dp_z \text{ states}$$

or

$$dN = \frac{V}{(2\pi\hbar)^3} d^3p . \tag{6.1.3}$$

For photons holds

$$dN = \frac{V}{(2\pi\hbar)^3} d^3k . \tag{6.1.4}$$

If there are two particles we have

$$dN = \frac{V}{(2\pi\hbar)^3} d^3p_1 \frac{V}{(2\pi\hbar)^3} d^3p_2 . \tag{6.1.5}$$

The cross section is determined mainly by the S matrix element S_{fi}. As mentioned in section 5.3 this element is the amplitude with which the envisaged final state

$|\Phi_f\rangle$ is present in the asymptotic state $\lim_{t\to\infty}|\Psi(t)\rangle \equiv \boldsymbol{S}|\Phi_i\rangle$. We know from quantum mechanics that the absolute square of this amplitude is just the probability for the existence of the final state $|\Phi_f\rangle$. Or, in other words, for particles with initial momenta $\underline{p}_1$ and $\underline{p}_2$ the quantity $\left|S_{fi}\left(\underline{p}_1', \underline{p}_2', \underline{p}_1, \underline{p}_2\right)\right|^2$ is the probability to reach the final state with momenta $\underline{p}_1'$ and $\underline{p}_2'$. Consequently, the expression

$$\left|S_{fi}\left(\underline{p}_1', \underline{p}_2', \underline{p}_1, \underline{p}_2\right)\right|^2 dN = \left|S_{fi}\left(\underline{p}_1', \underline{p}_2', \underline{p}_1, \underline{p}_2\right)\right|^2 V\frac{d^3p_1'}{(2\pi\hbar)^3} V\frac{d^3p_2'}{(2\pi\hbar)^3} \tag{6.1.6}$$

denotes the number of states which are reached in the momentum areas d^3p_1' and d^3p_2' near the momenta $\vec{p}_1'$ and $\vec{p}_2'$ starting from the initial state $(\vec{p}_1, \vec{p}_2)$ owing to the electromagnetic interaction.

We now deal with $|S_{fi}|^2$ and investigate how the box normalization can be used to transform the square of the δ-function in $(5.6.10/14)$. We look into the one-dimensional integral $\int_{-\infty}^{\infty} e^{\mathrm{i}\Delta px/\hbar} = 2\pi\hbar\delta(\Delta p)$ (c.f. $(3.2.25)$) and reduce the integrating sphere to $-L/2$ up to $+L/2$ with $L \gg \hbar/\Delta p$

$$\begin{aligned} 2\pi\hbar\delta(\Delta p) &= \int_{-\infty}^{\infty} e^{\mathrm{i}\Delta px/\hbar}dx \simeq \int_{-L/2}^{L/2} e^{\mathrm{i}\Delta px/\hbar}dx \\ &= \frac{\hbar}{\mathrm{i}\Delta p} e^{\mathrm{i}\Delta px/\hbar}\Big|_{-L/2}^{+L/2} = \frac{2\hbar\sin(\Delta pL/2\hbar)}{\Delta p}. \end{aligned} \tag{6.1.7}$$

We form the square of the expression $(6.1.7)$ like this

$$\left(\frac{2\hbar\sin(\Delta pL/2\hbar)}{\Delta p}\right)^2 \tag{6.1.8}$$

and consider $(6.1.8)$ as a function depending on Δp. It shows a peak at $\Delta p = 0$ with the height L^2 and zeros at $\pm\frac{2\pi\hbar}{L} \equiv \Delta p_{\text{zero}}$. Outside this region the oscillating function decreases rapidly. The area under this peak is nearly triangular and can be approximated by

$$\frac{1}{2}2|\Delta p_{\text{zero}}|\cdot L^2 = \frac{1}{2}\cdot\frac{2\cdot 2\pi\hbar}{L}L^2 = 2\pi\hbar L\,. \tag{6.1.9}$$

Therefore, it is not surprising that

$$\int_{-\infty}^{\infty}\left(\frac{2\hbar\sin(\Delta pL/2\hbar)}{\Delta p}\right)^2 d\Delta p = 2\pi\hbar L \tag{6.1.10}$$

holds exactly, which is proven by Greiner and Reinhardt, 1994, p. 93. On the other hand, using the definition of the δ-function we can write

$$\int_{-\infty}^{\infty} 2\pi\hbar L\delta(\Delta p)d\Delta p = 2\pi\hbar L . \qquad (6.1.11)$$

The integrands of $(6.1.10)$ and $(6.1.11)$ are equivalent

$$\left(\frac{2\hbar\sin(\Delta pL/2\hbar)}{\Delta p}\right)^2 \leftrightarrow 2\pi\hbar L\delta(\Delta p). \qquad (6.1.12)$$

Referring to $(6.1.7)$ we replace the function on the left hand side of $(6.1.12)$ like this

$$\frac{2\hbar\sin(\Delta pL/2\hbar)}{\Delta p} = \int_{-L/2}^{+L/2} e^{\mathrm{i}\Delta px/\hbar}dx \simeq \int_{-\infty}^{+\infty} e^{\mathrm{i}\Delta px/\hbar}dx \simeq 2\pi\hbar\delta(\Delta p) \qquad (6.1.13)$$

$$\text{resulting in } \left(2\pi\hbar\delta(\Delta p)\right)^2 \leftrightarrow 2\pi\hbar L\delta(\Delta p)$$
$$\text{or } \left(\delta(\Delta p)\right)^2 \leftrightarrow \frac{L}{2\pi\hbar}\delta(\Delta p) \qquad (6.1.14)$$

$$\text{and } \left(\delta(\Delta p_1)\delta(\Delta p_2)\delta(\Delta p_3)\right)^2 \leftrightarrow \left(\frac{L}{2\pi\hbar}\right)^3 \delta(\Delta p_1)\delta(\Delta p_2)\delta(\Delta p_3)$$
$$\text{or } \left(\delta^3(\delta\vec{p})\right)^2 \leftrightarrow \frac{V}{(2\pi\hbar)^3}\delta^3(\delta\vec{p}). \qquad (6.1.15)$$

We go to four dimensions and for the time we give the range *T* or we say that the scattering process lasts for the time *T*, i.e. the zero’th coordinate $x^0 = ct$ carries on cT . Instead of $(6.1.14)$ we obtain

$$\left(\delta(\Delta p_0)\right)^2 \leftrightarrow \frac{cT}{2\pi\hbar}\delta(\Delta p_0) \qquad (6.1.16)$$

and in four dimensions we have

$$\left(\delta^4(\Delta \underline{p})\right)^2 \leftrightarrow \frac{VTc}{(2\pi\hbar)^4}\delta^4(\Delta \underline{p}). \qquad (6.1.17)$$

Owing to $(5.6.14)$ we write

$$|S_{fi}|^2 = \left(\frac{e}{\hbar V}\right)^4$$
$$\cdot\frac{(2\pi)^8 m^4c^8}{E(\vec{p}_1)E(\vec{p}_2)E(\vec{p}_1')E(\vec{p}_2')}\hbar^8\left(\delta^4\left(\underline{p}_1'+\underline{p}_2'-\underline{p}_1-\underline{p}_2\right)\right)^2\left(\frac{\mathcal{E}'}{\mathcal{E}}\right)^8|M_{fi}|^2 \qquad (6.1.18)$$

with M_{fi}

$$= \sum_{\mu,\nu=0}^{3} [\underline{w}_{r_{2'}}^{\dagger}\left(\underline{p}_2'\right)\underline{\underline{\alpha}}_{\mu} g_{\mu\mu} \underline{w}_{r_2}\left(\underline{p}_2\right) D_F^{\mu\nu}\left(\frac{\underline{p}_1' - \underline{p}_1}{\hbar}\right) \underline{w}_{r_1'}^{\dagger}\left(\underline{p}_1'\right)\underline{\underline{\alpha}}_{\nu} g_{\nu\nu} \underline{w}_{r_1}\left(\underline{p}_1\right) \qquad (6.1.19)$$

$$- \underline{w}_{r_{2'}}^{\dagger}\left(\underline{p}_2'\right)\underline{\underline{\alpha}}_{\mu} g_{\mu\mu} \underline{w}_{r_1}\left(\underline{p}_1\right) D_F^{\mu\nu}\left(\frac{\underline{p}_1' - \underline{p}_2}{\hbar}\right) \underline{w}_{r_1'}^{\dagger}\left(\underline{p}_1'\right)\underline{\underline{\alpha}}_{\nu} g_{\nu\nu} \underline{w}_{r_2}\left(\underline{p}_2\right)]$$

or using $(6.1.17)$

$$\left|S_{fi}\right|^2 = \left(\frac{e}{\hbar V}\right)^4$$

$$\cdot \frac{(2\pi)^8 m^4 c^8}{E\left(\vec{p}_1\right)E\left(\vec{p}_2\right)E\left(\vec{p}_{1'}\right)E\left(\vec{p}_{2'}\right)} \frac{VTc\hbar^8}{(2\pi\hbar)^4} \delta^4\left(\underline{p}_1' + \underline{p}_2' - \underline{p}_1 - \underline{p}_2\right)\left(\frac{\mathcal{E}'}{\mathcal{E}}\right)^8 \left|M_{fi}\right|^2 . \qquad (6.1.20)$$

We will calculate the scattering cross section in the centre-of-mass frame of both electrons. In this system holds

$$\begin{aligned} &\vec{p}_1 + \vec{p}_2 = \vec{p}_1' + \vec{p}_2' = 0 \quad \text{and thus} \\ &\vec{p}_1' = -\vec{p}_2', \quad \vec{p}_1 = -\vec{p}_2 \quad \text{and} \\ &E\left(\vec{p}_1'\right) = \sqrt{m^2c^4 + \vec{p}_1'^2 c^2} = \sqrt{m^2c^4 + \vec{p}_2'^2 c^2} = E\left(\vec{p}_2'\right) \quad \text{and} \\ &E\left(\vec{p}_1\right) = E\left(\vec{p}_2\right). \end{aligned} \qquad (6.1.21)$$

With equation $(5.6.13)$ we obtain

$$E \equiv E\left(\vec{p}_1\right) = E\left(\vec{p}_2\right) = E\left(\vec{p}_1'\right) = E\left(\vec{p}_2'\right) \equiv E' . \qquad (6.1.22)$$

As mentioned at the beginning of this chapter the current J_{inc} of incident particles is needed for the calculation of the cross section. We calculate the incoming current of one electron. According to $(4.3.3)$ the i'th component is calculated like this

$$J^{(i)}{}_{\text{inc}} = c\underline{\psi}^{\dagger}\underline{\underline{\alpha}}_i\underline{\psi} \qquad (i = 1,2,3). \qquad (6.1.23)$$

Choosing the following components for the incoming particle $p_1^{(1)} = p_1^{(2)} = 0$ we let the wave go in the *z*-direction and determine the current $J_{\text{inc}}^{(3)}$ in this direction. First we take the variant $\underline{\varphi}^{(r=1)}\left(\vec{x},t\right)$ (see $(3.2.9)$) for ψ in $(6.1.23)$ as follows

$$J_{\text{inc}}^{(3)} = c\underline{\varphi}^{(r=1),\dagger}\left(\vec{x},t\right)\underline{\underline{\alpha}}_3\underline{\varphi}^{(r=1)}\left(\vec{x},t\right). \qquad (6.1.24)$$

Using $(3.2.9)$ we write

$$J_{\text{inc}}^{(3)} = c\frac{E(\vec{p})+mc^2}{2VE(\vec{p})}$$

$$\cdot\left(1,0,\frac{cp^{(3)}}{E(\vec{p})+mc^2},0\right)\begin{pmatrix}0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{pmatrix}\begin{pmatrix}1\\0\\\frac{cp^{(3)}}{E(\vec{p})+mc^2}\\0\end{pmatrix}$$

$$= c\frac{E(\vec{p})+mc^2}{2VE(\vec{p})}\left(1,0,\frac{cp^{(3)}}{E(\vec{p})+mc^2},0\right)\begin{pmatrix}\frac{cp^{(3)}}{E(\vec{p})+mc^2}\\0\\1\\0\end{pmatrix}$$

$$= c\frac{E(\vec{p})+mc^2}{2VE(\vec{p})}2\frac{cp^{(3)}}{E(\vec{p})+mc^2} = \frac{c^2p^{(3)}}{E(\vec{p})V}. \tag{6.1.25}$$

With $\underline{\varphi}^{(r=2)}(\vec{x},t)$ one gets the same results. In $(6.1.25)$ we insert well-known relativistic expressions for p and E

$$J_{\text{inc}}^{(3)} = \frac{mv^{(3)}}{\sqrt{1-v^2/c^2}}\frac{\sqrt{1-v^2/c^2}}{mc^2}\frac{c^2}{V} = \frac{|v|}{V}. \tag{6.1.26}$$

If there are two colliding particles with velocities $\vec{v}_1$ and $\vec{v}_2$ in $(6.1.26)$ the velocity $|\vec{v}|$ has to be replaced by the magnitude of the relative velocity $|\vec{v}_1-\vec{v}_2|$. (It can be larger than the velocity of light. However, it is not in contradiction with special relativity because $\vec{v}_2$ and $\vec{\text{v}}_2$ are velocities relative to the inertial system and $|\vec{v}_1-\vec{v}_2|$ is not the relativistic velocity of one particle in the system of the other one)

$$J_{\text{inc}} = \frac{|\vec{v}_1-\vec{v}_2|}{V}. \tag{6.1.27}$$

Owing to $(6.1.21)$ in the centre of mass system there holds

$$J_{\text{inc}} = \frac{|2\vec{v}_1|}{V}. \tag{6.1.28}$$

Now we are able to formulate the differential scattering cross section of electrons. Due to $(6.1.6)$, $(6.1.20)$, $(6.1.22)$ and the definition of T in $(6.1.16)$, the number of states which are reached in the momentum areas d^3p_1' and d^3p_2' near the momenta $\vec{p}_1'$ and $\vec{p}_2'$ per time unit is

$$\begin{aligned}&\left|S_{fi}\left(\underline{p}_1',r_1',\underline{p}_2',r_2',\underline{p}_1,r_1,\underline{p}_2,r_2\right)\right|^2 dN/T\\&=\left(\frac{2\pi e}{VE}\right)^4 m^4c^8VTc\delta^4\left(\underline{p}_1'+\underline{p}_2'-\underline{p}_1-\underline{p}_2\right)\left(\frac{\mathcal{E}'}{\mathcal{E}}\right)^8\\&\cdot\left|M_{fi}\left(\underline{p}_1',r_1',\underline{p}_2',r_2',\underline{p}_1,r_1,\underline{p}_2,r_2\right)\right|^2 V\frac{d^3p_1'}{(2\pi\hbar)^3}V\frac{d^3p_2'}{(2\pi\hbar)^3}\frac{1}{T}.\end{aligned}\tag{6.1.29}$$

The differential cross section $d\tilde{\sigma}$ for the transition from the state $\left(\underline{p}_1,r_1,\underline{p}_2,r_2\right)$ to states within $\left(\underline{p}_1',r_1',\underline{p}_2',r_2'\right)$ and $\left(\underline{p}_1'+d^3p_1',r_1',\underline{p}_2'+d^3p_2',r_2'\right)$ is the transition rate $(6.1.29)$ per initial current $(6.1.28)$. (It can be interpreted as a virtual area which is held in the current of the incidental particles and which catches particles of the state $\left(\underline{p}_1,r_1,\underline{p}_2,r_2\right)$ and brings them to the states $\left(\underline{p}_1',r_1',\underline{p}_2',r_2'\right)$.)

$$\begin{aligned}d\tilde{\sigma}&=\left(\frac{2\pi e}{VE}\right)^4 m^4c^9\delta^4\left(\underline{p}_1'+\underline{p}_2'-\underline{p}_1-\underline{p}_2\right)\left|M_{fi}\left(\underline{p}_1',r_1',\underline{p}_2',r_2',\underline{p}_1,r_1,\underline{p}_2,r_2\right)\right|^2\\&\cdot\left(\frac{\mathcal{E}'}{\mathcal{E}}\right)^8\frac{V^3}{(2\pi\hbar)^6}d\left|p_1'\right|\left|\vec{p}_1'\right|^2 d\Omega_1'\cdot d^3p_2'\frac{V}{2\left|v_1\right|},\end{aligned}\tag{6.1.30}$$

where we have written the spatial differential d^3p_1' as $d\left|p_1'\right|\cdot\left|\vec{p}_1'\right|^2 d\Omega_1'$ with the solid angle $d\Omega_1'$ in which the scattered particles move.

Now, we admit all momenta $\vec{p}_2'$ and all magnitudes $\left|\vec{p}_1'\right|$ of $\vec{p}_1'$. I.e. we are interested in particles of type *1*' which pass through the solid angle $d\Omega_1'$ and we integrate over all components of $\vec{p}_2'$ and over $\left|\vec{p}_1'\right|$

$$\begin{aligned}d\sigma&=d\Omega_1'\left(\frac{em}{E}\right)^4\frac{c^9}{(2\pi)^2 2\left|v_1\right|}\int_0^\infty\frac{d\left|\vec{p}_1'\right|\left|\vec{p}_1'\right|^2}{2E'}\int\frac{d^3p_2'}{2E'}\delta^4\left(\underline{p}_1'+\underline{p}_2'-\underline{p}_1-\underline{p}_2\right)\\&\cdot\left(\frac{\mathcal{E}'}{\mathcal{E}}\right)^8\left|M_{fi}\left(\underline{p}_1',r_1',\underline{p}_2',r_2',\underline{p}_1,r_1,\underline{p}_2,r_2\right)\right|^2\frac{(2E')^2}{\hbar^6}.\end{aligned}\tag{6.1.31}$$

In the integral we have introduced the cancelling factors $\frac{1}{4E'^2}$ and $4E'^2$. We show that it is possible to write $\frac{1}{2E'}$ as an integral over a delta function with the zero-component p'^0 of the four-dimensional momentum $\underline{p}'$ as integration variable like this

$$\begin{aligned}&c\int_0^\infty dp'^0\delta\left(\left(\underline{p}'\right)^2c^2-m^2c^4\right)=c\int_0^\infty dp'^0\delta\left(\left(p'^0\right)^2c^2-\vec{p}'^2c^2-m^2c^4\right)\\&=c\int_0^\infty dp'^0\delta\left(\left(p'^0\right)^2c^2-E'^2\right),\end{aligned}\tag{6.1.32}$$

where we have used $(3.2.2)$. The integrand of $(6.1.32)$ has the positive root $p'^0 = E'/c$. We use now the well-known mathematical formula

$$\int dx\delta\left(f(x)\right)F(x) = \sum_k F(x_k)/\left|\frac{df}{dx}\right|_{x_k}, \qquad (6.1.33)$$

where x_k is a root of $f(x)$ within the interval of integration. We apply $(6.1.33)$ to $(6.1.32)$ where the function $F(x)$ is equal to 1.

$$c\int_0^\infty dp'^0\delta\left(\left(p'^0\right)^2 c^2 - E'^2\right) = \frac{c}{\left|\frac{d\left(\left(p'^0\right)^2 c^2 - E'^2\right)}{dp'^0}\right|_{p'^0=E'/c}} = \frac{1}{2E'}$$

$$\text{or } c\int_{-\infty}^{\infty} dp'^0\delta\left(\underline{p}'^2c^2 - m^2c^4\right)\Theta\left(p'^0\right) = \frac{1}{2E'} \qquad (6.1.34)$$

$$\text{with } \left\{\begin{matrix} \Theta\left(p'^0\right) = 1 & \text{for } p'^0 > 0 \\ \Theta\left(p'^0\right) = 0 & \text{for } p'^0 < 0 \end{matrix}\right\}.$$

We pick out the integrals of $(6.1.31)$ ad insert $(6.1.34)$ for one factor $\frac{1}{2E'}$

$$\begin{aligned} I &= \int_0^\infty \frac{d|p_1'||p_1'|^2}{2E'}\int\frac{d^3p_2'}{2E'}\delta^4\left(\underline{p}_1' + \underline{p}_2' - \underline{p}_1 - \underline{p}_2\right) \\ &\quad\cdot\left|M_{fi}\left(\underline{p}_1', r_1', \underline{p}_2', r_2', \underline{p}_1, r_1, \underline{p}_2, r_2\right)\right|^2 \cdot 2E'2E'/\hbar^6 \\ &= \int_0^\infty \frac{d|p_1'||p_1'|^2}{2E'}\int_{-\infty}^{\infty} d^3p_2' c\int_{-\infty}^{\infty} dp_2'^0 \\ &\quad\cdot\delta\left(\left(\underline{p}_2'\right)^2 c^2 - m^2c^4\right)\Theta\left(p_2'^0\right)\delta^4\left(\underline{p}_1' + \underline{p}_2' - \underline{p}_1 - \underline{p}_2\right)\cdot f\left(\underline{p}_1', \underline{p}_2', \cdots\right) \\ &\quad\text{with } f\left(\underline{p}_1', \underline{p}_2', \cdots\right) = \left|M_{fi}\left(\underline{p}_1', r_1', \underline{p}_2', r_2', \underline{p}_1, r_1, \underline{p}_2, r_2\right)\right|^2 \cdot \left(2E'\right)^2/\hbar^6. \end{aligned} \qquad (6.1.35)$$

We carry out the integration $\int d^3p_2'\int_{-\infty}^{\infty} dp_2'^0 \equiv \int d^4p_2'$ .The term $\delta^4\left(\underline{p}_1' + \underline{p}_2' - \underline{p}_1 - \underline{p}_2\right)$ in the integrand of $(6.1.35)$ causes $\underline{p}_2' = \underline{p}_1 + \underline{p}_2 - \underline{p}_1'$ and $p_2'^0 = \frac{2E}{c} - \frac{E'}{c}$

and

$$\begin{aligned} I &= \int_0^\infty \frac{d|\vec{p}_1'||\vec{p}_1'|^2}{2E'}c\delta\left(\left(\underline{p}_1 + \underline{p}_2 - \underline{p}_1'\right)^2 c^2 - m^2c^4\right) \\ &\quad\cdot\Theta\left(\frac{2E}{c} - \frac{E'}{c}\right)\cdot f\left(\underline{p}_1', \underline{p}_2' = \underline{p}_1 + \underline{p}_2 - \underline{p}_1', \cdots\right). \end{aligned} \qquad (6.1.36)$$

Of course $0 \le E' \le 2E$ holds. Furthermore,

$$\begin{aligned}
&\left(\underline{p}_1+\underline{p}_2-\underline{p}_1'\right)^2 c^2-m^2c^4\\
&=\left(\underline{p}_1+\underline{p}_2\right)^2 c^2-2\left(\underline{p}_1+\underline{p}_2\right)\cdot\underline{p}_1'c^2+\underline{p}_1'^2c\ \ -m^2c^4\\
&=\left(\underline{p}_1+\underline{p}_2\right)^2 c^2-2\left(\underline{p}_1+\underline{p}_2\right)\cdot\underline{p}_1'c^2+\frac{E'^2c^2}{c^2}-\vec{p}_1'^2c^2-m^2c^4\\
&=\left(\underline{p}_1+\underline{p}_2\right)^2 c^2-2\left(\underline{p}_1+\underline{p}_2\right)\cdot\underline{p}_1'c^2+E'^2-E'^2+m^2c^4-m^2c^4\\
&=\left(\underline{p}_1+\underline{p}_2\right)^2 c^2-2\left(\underline{p}_1+\underline{p}_2\right)\cdot\underline{p}_1'c^2
\end{aligned}\tag{6.1.37}$$

where we have used $(3.2.2)$ and $(3.4.8)$. Because of $\left|\vec{p}_1'\right| d\left|\vec{p}_1'\right| c^2=E'dE'$ we have

$$\begin{aligned}
I=&\int_0^{2E}\frac{\left|\vec{p}_1'\right|E'dE'}{2E'c^2}c\delta\left(\left(\underline{p}_1+\underline{p}_2\right)^2c^2-2\left(\underline{p}_1+\underline{p}_2\right)\underline{p}_1'c^2\right)\\
&\cdot f\left(\underline{p}_1',\underline{p}_2'=\underline{p}_1+\underline{p}_2-\underline{p}_1',\cdots\right)\\
=&\frac{1}{2c}\int_0^{2E}\left|\vec{p}_1'\right|dE'\delta\left(\left(\frac{2E}{c}\right)^2c^2-2c^2\left(\frac{2E}{c}\right)\frac{E'}{c}+0\right)\\
&\cdot f\left(\underline{p}_1',\underline{p}_2'=\underline{p}_1+\underline{p}_2-\underline{p}_1',\cdots\right).
\end{aligned}\tag{6.1.38}$$

For the argument of the δ-function we have written only the zero'th component of the $\underline{p}$'s. The other vanish due to $(6.1.21)$. Applying $(6.1.33)$ to $(6.1.38)$ results in

$$I=\frac{\left|\vec{p}_1'\right|}{2c\left|-4E\right|}f\left(\underline{p}_1',\underline{p}_2'=\underline{p}_1+\underline{p}_2-\underline{p}_1',\cdots\right)\Big|_{E'=E}\tag{6.1.39}$$

which we insert in $(6.1.31)$

$$\begin{aligned}
d\sigma=&d\Omega'\left(\frac{me}{E}\right)^4\frac{c^9}{(2\pi)^2\,2\left|\vec{v}_1\right|}\frac{\left|\vec{p}_1'\right|4E^2}{4E2c\hbar^6}\left(\frac{\mathcal{E}'}{\mathcal{E}}\right)^8\\
&\cdot\left|M_{fi}\left(\underline{p}_1',r_1',\underline{p}_2',r_2',\underline{p}_1,r_1,\underline{p}_2,r_2\right)\right|^2_{\substack{\underline{p}_2'=\underline{p}_1+\underline{p}_2-\underline{p}_1'\\ E'=E}}.
\end{aligned}\tag{6.1.40}$$

In most electron scattering experiments unpolarized electrons collide and the polarization of the scattered electrons isn't measured. Accordingly the scattering cross sections for the incoming electrons with the possible spin directions relative to $\vec{p}$ (lengthwise and opposite) have to be averaged (summed up for both particles and divided by 4). Because both spin directions of the outgoing electrons are accepted, the corresponding cross sections have to be added up. We show that the index r in $\underline{w}_r\left(\underline{p}\right)$ denotes the spin direction relative to $\vec{p}$. As in $(6.1.25)$ in $\underline{w}_r\left(\underline{p}\right)$, $(3.2.19)$, we limit $\vec{p}$ to the z-direction as follows

$$\underline{w}_{r=1,z}\left(\underline{p}\right)=\sqrt{\frac{E+mc^2}{2mc^2}}\begin{pmatrix}1\\0\\\dfrac{cp^{(3)}}{E+mc^2}\\0\end{pmatrix},$$
$$\underline{w}_{r=2,z}\left(\underline{p}\right)=\sqrt{\frac{E+mc^2}{2mc^2}}\begin{pmatrix}0\\1\\0\\\dfrac{-cp^{(3)}}{E+mc^2}\end{pmatrix} \tag{6.1.41}$$

As shown by Pfeifer, 2004, p.35, the helicity operator

$$\underline{\underline{S}}_{p,z}=\frac{\hbar}{2}\begin{pmatrix}1&&&0\\&-1&&\\&&1&\\0&&&-1\end{pmatrix} \tag{6.1.42}$$

is an eigenoperator of $\underline{w}_z\left(\underline{p},r\right)$ like this

$$\underline{\underline{S}}_{p,z}\ \underline{w}_{r=1,z}\left(\underline{p}\right)=+\frac{\hbar}{2}\underline{w}_{r=1,z}\left(\underline{p}\right)\ \text{ and }$$
$$\underline{\underline{S}}_{p,z}\ \underline{w}_{r=2,z}\left(\underline{p}\right)=-\frac{\hbar}{2}\underline{w}_{r=2,z}\left(\underline{p}\right). \tag{6.1.43}$$

Thus, for $r=1$ the spin looks forwards and for $r=2$ backwards. We denote the differential scattering cross section for unpolarized electrons by

$$\overline{\frac{d\sigma}{d\Omega_1'}}=\frac{e^4m^4c^8}{\left(2\pi\right)^2E^2 4v_1}\frac{\left|\vec{p}_1'\right|}{E\hbar^6}\left(\frac{\mathcal{E}'}{\mathcal{E}}\right)^8\overline{\left|M^{fi}\left(\underline{p}_1',r_1',\underline{p}_2',r_2',\underline{p}_1,r_1,\underline{p}_2,r_2\right)\right|^2}\Bigg|_{\substack{\underline{p}_2'=\underline{p}_1+\underline{p}_2-\underline{p}_1'\\E'=E}} \tag{6.1.44}$$

with (c.f. $(6.1.19)$ and $(5.6.9)$)

$$\begin{aligned}&\overline{\left|M_{fi}\left(\underline{p}_1',r_1',\underline{p}_2',r_2',\underline{p}_1,r_1,\underline{p}_2,r_2\right)\right|^2}\simeq\left(\frac{\hbar}{\varepsilon_0 c}\right)^2\frac{1}{4}\\&\cdot\sum_{r_1',r_2'=1}^{2}\sum_{r_1,r_2=1}^{2}|\sum_{\mu=0}^{3}g_{\mu\mu}[\mathrm{w}_{r_2'}^{\dagger}\left(\underline{p}_2'\right)\underline{\underline{\alpha}}_\mu g_{\mu\mu}\underline{w}_{r_2}\left(\underline{p}_2\right)\frac{\hbar^2}{\left(\underline{p}_1-\underline{p}_1'\right)^2}\underline{w}_{r_1'}^{\dagger}\left(\underline{p}_1'\right)\underline{\underline{\alpha}}_\mu g_{\mu\mu}\underline{w}_{r_1}\left(\underline{p}_1\right)\\&\qquad-\mathrm{w}_{r_2'}^{\dagger}\left(\underline{p}_2'\right)\underline{\underline{\alpha}}_\mu g_{\mu\mu}\underline{w}_{r_1}\left(\underline{p}_1\right)\frac{\hbar^2}{\left(\underline{p}_1-\underline{p}_2'\right)^2}\underline{w}_{r_1'}^{\dagger}\left(\underline{p}_1'\right)\underline{\underline{\alpha}}_\mu g_{\mu\mu}\underline{w}_{r_2}\left(\underline{p}_2\right)]|^2\\&=\overline{T}_1+\overline{T}_2.\end{aligned} \tag{6.1.45}$$

In the third line we have replaced $\left(\underline{p}_1'-\underline{p}_2\right)^2$ by $\left(\underline{p}_1-\underline{p}_2'\right)^2$ due to the δ^4-function in $(6.1.30)$. The first term in $\left|\overline{M}_{fi}\right|^2$ reads

$$\bar{T}_1 = \left(\frac{\hbar}{\varepsilon_0 c}\right)^2 \frac{1}{4}$$

$$\cdot \sum_{r_1',r_2'=1}^{2} \sum_{r_1,r_2=1}^{2} \left(\frac{\hbar}{\underline{p}_1 - \underline{p}_1'}\right)^4 \sum_{\mu.\lambda=0}^{3} [\underline{w}_{r_2'}^{\dagger}(p_2')\underline{\underline{\alpha}}_{\mu} g_{\mu\mu} \underline{w}_{r_2}(\underline{p}_2) \underline{w}_{r_1'}^{\dagger}(\underline{p}_1')\underline{\underline{\alpha}}_{\mu} \underline{w}_{r_1}(\underline{p}_1)] \qquad (6.1.46)$$

$$\cdot [\underline{w}_{r_2'}^{\dagger}(p_2')\underline{\underline{\alpha}}_{\lambda} g_{\lambda\lambda} \underline{w}_{r_2}(\underline{p}_2) \underline{w}_{r_1'}^{\dagger}(\underline{p}_1')\underline{\underline{\alpha}}_{\lambda} \underline{w}_{r_1}(\underline{p}_1)]^*.$$

Because $\underline{w}_{r'}^{\dagger}(\underline{p}')\underline{\underline{\alpha}}_{\kappa}\underline{w}_r(\underline{p})$ is a number we can write $\left(\underline{w}_{r'}^{\dagger}(\underline{p}')\underline{\underline{\alpha}}_{\kappa}\underline{w}_r(\underline{p})\right)^*$ $= \left(\underline{w}_{r'}^{\dagger}(\underline{p}')\underline{\underline{\alpha}}_{\kappa}\underline{w}_r(\underline{p})\right)^{\dagger}$. We use the mathematical relation

$$\left(\underline{w}_{r'}^{\dagger}(\underline{p}')\underline{\underline{\alpha}}_{\kappa}\underline{w}_r(\underline{p})\right)^{\dagger} = \underline{w}_r^{\dagger}(\underline{p})\underline{\underline{\alpha}}_{\kappa}\underline{w}_{r'}(\underline{p}') \qquad (6.1.47)$$

where $\underline{\underline{\alpha}}_{\kappa} = \underline{\underline{\alpha}}_{\kappa}^{\dagger}$ holds. We rearrange the brackets in $(6.1.46)$ like this

$$\bar{T}_1 = \left(\frac{\hbar}{\varepsilon_0 c}\right)^2 \frac{1}{4}$$

$$\cdot \sum_{r_1',r_2'=1}^{2} \sum_{r_1,r_2=1}^{2} \left(\frac{\hbar}{\underline{p}_1 - \underline{p}_1'}\right)^4 \sum_{\mu.\lambda=0}^{3} [\underline{w}_{r_2'}^{\dagger}(p_2')\underline{\underline{\alpha}}_{\mu} g_{\mu\mu} \underline{w}_{r_2}(\underline{p}_2) \underline{w}_{r_1'}^{\dagger}(\underline{p}_1')\underline{\underline{\alpha}}_{\mu} \underline{w}_{r_1}(\underline{p}_1) \qquad (6.1.48)$$

$$\cdot \underline{w}_{r_1}{}^{\dagger}(\underline{p}_1)\underline{\underline{\alpha}}_{\lambda} \underline{w}_{r_1'}(\underline{p}_1') \underline{w}_{r_2}^{\dagger}(\underline{p}_2)\underline{\underline{\alpha}}_{\lambda} g_{\lambda\lambda} \underline{w}_{r_2'}(p_2')].$$

Due to $(3.4.9)$ and to $\underline{\underline{\beta}}^2 = \underline{\underline{1}}$ we can write

$$\underline{w}_r{}^{\dagger}(\underline{p}) = \bar{\underline{w}}_r(\underline{p})\underline{\underline{\beta}}. \qquad (6.1.48a)$$

Now we deal with inner products in $(6.1.48)$, write them explicitly and apply $(3.4.13)$ twice

$$\begin{aligned}
&\sum_{r_1'}\sum_{r_1} \underline{w}_{r_1'}^{\dagger}\left(\underline{p}_1'\right)\underline{\underline{\alpha}}_{\mu}\underline{w}_{r_1}\left(\underline{p}_1\right)\underline{w}_{r_1}^{\dagger}\left(\underline{p}_1\right)\underline{\underline{\alpha}}_{\lambda}\underline{w}_{r_1'}\left(\underline{p}_1'\right)\\
&=\sum_{r_1'}\underline{\bar{w}}_{r_1'}\left(p_1'\right)\underline{\underline{\beta\alpha}}_{\mu}\left(\sum_{r_1}\underline{w}_{r_1}\left(\underline{p}_1\right)\underline{\bar{w}}_{r_1}\left(\underline{p}_1\right)\right)\underline{\underline{\beta\alpha}}_{\lambda}\underline{w}_{r_1'}\left(\underline{p}_1'\right)\\
&=\sum_{\alpha\beta\gamma\delta}\sum_{r_1'}\underline{\bar{w}}_{r_1',\alpha}\left(p_1'\right)\left(\underline{\underline{\beta\alpha}}_{\mu}\right)_{\alpha\beta}\left(\sum_{r_1}\underline{w}_{r_{1,\beta}}\left(\underline{p}_1\right)\underline{\bar{w}}_{r_{1,\gamma}}\left(\underline{p}_1\right)\right)\left(\underline{\underline{\beta\alpha}}_{\lambda}\right)_{\gamma\delta}\underline{w}_{r_1',\delta}\left(\underline{p}_1'\right)\\
&=\sum_{\alpha\delta}\sum_{r_1'}\left(\underline{\underline{\beta\alpha}}_{\mu}\frac{c\underline{\underline{\not{p}}}_1+mc^2\underline{\underline{1}}}{2mc^2}\underline{\underline{\beta\alpha}}_{\lambda}\right)_{\alpha\delta}\underline{w}_{r_1',\delta}\left(\underline{p}_1'\right)\underline{\bar{w}}_{r_1',\alpha}\left(p_1'\right)\\
&=\sum_{\alpha\delta}\left(\underline{\underline{\beta\alpha}}_{\mu}\frac{c\underline{\underline{\not{p}}}_1+mc^2\underline{\underline{1}}}{2mc^2}\underline{\underline{\beta\alpha}}_{\lambda}\right)_{\alpha\delta}\left(\frac{c\underline{\underline{\not{p}}}_1'+mc^2\underline{\underline{1}}}{2mc^2}\right)_{\delta\alpha}\\
&=\mathrm{Tr}\left[\underline{\underline{\beta\alpha}}_{\mu}\frac{c\underline{\underline{\not{p}}}_1+mc^2\underline{\underline{1}}}{2mc^2}\underline{\underline{\beta\alpha}}_{\lambda}\frac{c\underline{\underline{\not{p}}}_1'+mc^2\underline{\underline{1}}}{2mc^2}\right].
\end{aligned}\tag{6.1.49}$$

The remaining factors in $(6.1.48)$ produce a similar trace like $(6.1.49)$ containing $\underline{\underline{\not{p}}}_2$ and $\underline{\underline{\not{p}}}_2'$. That is why $(6.1.48)$ results in

$$\begin{aligned}
\bar{T}_1=\left(\frac{\hbar}{2\varepsilon_0 c}\right)^2\left(\frac{\hbar}{\underline{p}_1-\underline{p}_1'}\right)^4\sum_{\mu\lambda}\mathrm{Tr}\left[\underline{\underline{\beta\alpha}}_{\mu}\frac{c\underline{\underline{\not{p}}}_1+mc^2\underline{\underline{1}}}{2mc^2}\underline{\underline{\beta\alpha}}_{\lambda}\frac{c\underline{\underline{\not{p}}}_1'+mc^2\underline{\underline{1}}}{2mc^2}\right]&\\
\cdot\mathrm{Tr}\left[\underline{\underline{\beta\alpha}}_{\mu}g_{\mu\mu}\frac{c\underline{\underline{\not{p}}}_2+mc^2\underline{\underline{1}}}{2mc^2}\underline{\underline{\beta\alpha}}_{\lambda}g_{\lambda\lambda}\frac{c\underline{\underline{\not{p}}}_2'+mc^2\underline{\underline{1}}}{2mc^2}\right]&.
\end{aligned}\tag{6.1.50}$$

The next term in $(6.1.45)$ is mixed. We treat it analogously to $(6.1.49)$

$$\bar{T}_2=\left(\frac{\hbar}{2\varepsilon_0 c}\right)^2\frac{\hbar^4}{\left(\underline{p}_1-\underline{p}_1'\right)^2\left(\underline{p}_1-\underline{p}_2'\right)^2}\sum_{\mu\lambda}\tilde{T}_2\quad\text{with}$$

$$\tilde{\mathrm{T}}_2 = \sum_{r_1',r_2'}\sum_{r_1,r_2}(\underline{w}^{\dagger}_{r_2'}\left(\underline{p}_2'\right)\underline{\underline{\alpha}}_{\mu}g_{\mu\mu}\underline{w}_{r_2}\left(\underline{p}_2\right)\underline{w}^{\dagger}_{r_1'}\left(\underline{p}_1'\right)\underline{\underline{\alpha}}_{\mu}\underline{w}_{r_1}\left(\underline{p}_1\right))$$
$$\cdot(\underline{w}^{\dagger}_{r_2'}\left(\underline{p}_2'\right)\underline{\underline{\alpha}}_{\lambda}g_{\lambda\lambda}\underline{w}_{r_1}\left(\underline{p}_1\right)\underline{w}^{\dagger}_{r_1'}\left(\underline{p}_1'\right)\underline{\underline{\alpha}}_{\mu}\underline{w}_{r_2}\left(\underline{p}_2\right))^*$$
$$= \sum_{r_1',r_2'}\sum_{r_1}(\underline{\overline{w}}_{r_1'}\left(\underline{p}_1'\right)\underline{\underline{\beta}}\underline{\underline{\alpha}}_{\mu}\underline{w}_{r_1}\left(\underline{p}_1\right))(\underline{\overline{w}}_{r_1}\left(\underline{p}_1\right)\underline{\underline{\beta}}\underline{\underline{\alpha}}_{\lambda}g_{\lambda\lambda}\underline{w}_{r_2'}\left(\underline{p}_2'\right)$$
$$\cdot\sum_{r_2}(\underline{\overline{w}}_{r_2'}\left(\underline{p}_2'\right)\underline{\underline{\beta}}\underline{\underline{\alpha}}_{\mu}g_{\mu\mu}\underline{w}_{r_2}\left(\underline{p}_2\right))(\underline{\overline{w}}_{r_2}\left(\underline{p}_2\right)\underline{\underline{\beta}}\underline{\underline{\alpha}}_{\lambda}\underline{w}_{r_1'}\left(\underline{p}_1'\right)$$
$$= \sum_{r_1',r_2'}\underline{\overline{w}}_{r_1'}\left(\underline{p}_1'\right)\underline{\underline{\beta}}\underline{\underline{\alpha}}_{\mu}\frac{c\underline{\underline{\not{p}}}_1 + mc^2\underline{\underline{1}}}{2mc^2}\underline{\underline{\beta}}\underline{\underline{\alpha}}_{\lambda}g_{\lambda\lambda}\underline{w}_{r_2'}\left(\underline{p}_2'\right)$$
$$\cdot\underline{\overline{w}}_{r_2'}\left(\underline{p}_2'\right)\underline{\underline{\beta}}\underline{\underline{\alpha}}_{\mu}g_{\mu\mu}\frac{c\underline{\underline{\not{p}}}_2 + mc^2\underline{\underline{1}}}{2mc^2}\underline{\underline{\beta}}\underline{\underline{\alpha}}_{\lambda}\underline{w}_{r_1'}\left(\underline{p}_1'\right)$$
$$= \sum_{\alpha\delta}\sum_{r_1'}\underline{\overline{w}}_{r_1',\alpha}(\left(\underline{p}_1'\right)\underline{\underline{\beta}}\underline{\underline{\alpha}}_{\mu}\frac{c\underline{\underline{\not{p}}}_1 + mc^2\underline{\underline{1}}}{2mc^2}\underline{\underline{\beta}}\underline{\underline{\alpha}}_{\lambda}g_{\lambda\lambda}\frac{c\underline{\underline{\not{p}}}_2' + mc^2\underline{\underline{1}}}{2mc^2}$$
$$\cdot\underline{\underline{\beta}}\underline{\underline{\alpha}}_{\mu}g_{\mu\mu}\frac{c\underline{\underline{\not{p}}}_2 + mc^2\underline{\underline{1}}}{2mc^2}\underline{\underline{\beta}}\underline{\underline{\alpha}}_{\lambda})_{\alpha\delta}\underline{w}_{r_1',\delta}\left(\underline{p}_1'\right)$$
$$= \mathrm{Tr}(\underline{\underline{\beta}}\underline{\underline{\alpha}}_{\mu}\frac{c\underline{\underline{\not{p}}}_1 + mc^2\underline{\underline{1}}}{2mc^2}\underline{\underline{\beta}}\underline{\underline{\alpha}}_{\lambda}g_{\lambda\lambda}\frac{c\underline{\underline{\not{p}}}_2' + mc^2\underline{\underline{1}}}{2mc^2}$$
$$\cdot\underline{\underline{\beta}}\underline{\underline{\alpha}}_{\mu}g_{\mu\mu}\frac{c\underline{\underline{\not{p}}}_2 + mc^2\underline{\underline{1}}}{2mc^2}\underline{\underline{\beta}}\underline{\underline{\alpha}}_{\lambda}\frac{c\underline{\underline{\not{p}}}_1' + mc^2\underline{\underline{1}}}{2mc^2}). \tag{6.1.51}$$

We insert $(6.1.50)$ and $(6.1.51)$ in $(6.1.45)$ like this

$$\overline{\left|M_{fi}\left(\underline{p}_1',r_1',\underline{p}_2',r_2',\underline{p}_1,r_1,\underline{p}_2,r_2\right)\right|^2} = \left(\frac{\hbar}{2\varepsilon_0 c}\right)^2$$
$$\cdot\sum_{\mu,\lambda}[\frac{\hbar^4}{\left(\underline{p}_1 - p_1'\right)^4}\mathrm{Tr}(\underline{\underline{\beta}}\underline{\underline{\alpha}}_{\mu}\frac{c\underline{\underline{\not{p}}}_1 + mc^2\underline{\underline{1}}}{2mc^2}\underline{\underline{\beta}}\underline{\underline{\alpha}}_{\lambda}\frac{c\underline{\underline{\not{p}}}_1' + mc^2\underline{\underline{1}}}{2mc^2})$$
$$\cdot\mathrm{Tr}(\underline{\underline{\beta}}\underline{\underline{\alpha}}_{\mu}g_{\mu\mu}\frac{c\underline{\underline{\not{p}}}_2 + mc^2\underline{\underline{1}}}{2mc^2}\underline{\underline{\beta}}\underline{\underline{\alpha}}_{\lambda}g_{\lambda\lambda}\frac{c\underline{\underline{\not{p}}}_2' + mc^2\underline{\underline{1}}}{2mc^2}) \tag{6.1.52}$$
$$-\frac{\hbar^4}{\left(\underline{p}_1 - p_1'\right)^2\left(\underline{p}_1 - p_2'\right)^2}\mathrm{Tr}(\underline{\underline{\beta}}\underline{\underline{\alpha}}_{\mu}\frac{c\underline{\underline{\not{p}}}_1 + mc^2\underline{\underline{1}}}{2mc^2}\underline{\underline{\beta}}\underline{\underline{\alpha}}_{\lambda}g_{\lambda\lambda}\frac{c\underline{\underline{\not{p}}}_2' + mc^2\underline{\underline{1}}}{2mc^2}$$
$$\cdot\underline{\underline{\beta}}\underline{\underline{\alpha}}_{\mu}g_{\mu\mu}\frac{c\underline{\underline{\not{p}}}_2 + mc^2\underline{\underline{1}}}{2mc^2}\underline{\underline{\beta}}\underline{\underline{\alpha}}_{\lambda}\frac{c\underline{\underline{\not{p}}}_1' + mc^2\underline{\underline{1}}}{2mc^2}) + \left(\underline{p}_1' \rightleftarrows \underline{p}_2'\right)].$$

by writing $\left(\underline{p}_1' \rightleftarrows \underline{p}_2'\right)$ we have abbreviated the terms that can be generated from those written down by exchanging $\underline{p}_1'$ and $\underline{p}_2'$, which can be seen by comparing the first line in $(6.1.45)$ with the second one.

The equations $(6.1.44)$ and $(6.1.52)$ constitute the differential scattering cross section for unpolarized electrons in the centre of mass system.

Now, we consider the ultra relativistic limit where $p^0 c = \sqrt{\vec{p}^2 c^2 + m^2 c^4} = E \gg mc^2$ for which reason we neglect $mc\underline{\underline{1}}$ with regard to $\underline{\underline{\not p}}$. We introduce the following practical definition

$$\underline{\underline{\gamma}}^0 \equiv \underline{\underline{\beta}}, \quad \underline{\underline{\gamma}}^i \equiv \underline{\underline{\beta}}\underline{\underline{\alpha}}_i \quad i = 1,2,3\,, \tag{6.1.53}$$

which we insert in $\bar{T}_2, (6.1.51)$, in the ultra relativistic limit

$$\begin{aligned}\bar{T}_{2,\mathrm{ur}} &= \left(\frac{\hbar}{2\varepsilon_0 c}\right)^2 \frac{1}{(2mc)^4} \frac{\hbar^4}{\left(\underline{p}_1 - \underline{p}_1'\right)^2 \left(\underline{p}_1 - \underline{p}_2'\right)^2} \\ &\cdot \sum_{\mu\lambda} \mathrm{Tr}\left(g_{\mu\mu} g_{\lambda\lambda} \underline{\underline{\gamma}}^\mu \underline{\underline{\not p}}_1 \underline{\underline{\gamma}}^\lambda \underline{\underline{\not p}}_2' \underline{\underline{\gamma}}^\mu \underline{\underline{\not p}}_2 \underline{\underline{\gamma}}^\lambda \underline{\underline{\not p}}_1'\right).\end{aligned} \tag{6.1.54}$$

Applying twice the well-known mathematical relation for square matrices $\underline{\underline{A}}$ and $\underline{\underline{B}}$

$$\mathrm{Tr}\left(\underline{\underline{A}}\underline{\underline{B}}\right) = \mathrm{Tr}\left(\underline{\underline{B}}\underline{\underline{A}}\right) \tag{6.1.55}$$

we obtain

$$\begin{aligned}\bar{T}_{2,\mathrm{ur}} &= \left(\frac{\hbar}{2\varepsilon_0 c}\right)^2 \frac{1}{(2mc)^4} \frac{\hbar^4}{\left(\underline{p}_1 - \underline{p}_1'\right)^2 \left(\underline{p}_1 - \underline{p}_2'\right)^2} \\ &\cdot \sum_{\mu\lambda} \mathrm{Tr}\left(g_{\mu\mu} g_{\lambda\lambda} \underline{\underline{\gamma}}^\lambda \underline{\underline{\not p}}_1' \underline{\underline{\gamma}}^\mu \underline{\underline{\not p}}_1 \underline{\underline{\gamma}}^\lambda \underline{\underline{\not p}}_2' \underline{\underline{\gamma}}^\mu \underline{\underline{\not p}}_2\right).\end{aligned} \tag{6.1.56}$$

In order to simplify this expression we have to do several reflections. First, the relation

$$\underline{\underline{\gamma}}^\mu \underline{\underline{\gamma}}^\nu + \underline{\underline{\gamma}}^\nu \underline{\underline{\gamma}}^\mu = 2 g_{\mu\nu} \underline{\underline{1}} \tag{6.1.57}$$

can be proven by calculation by hand starting from $\underline{\underline{\beta}}$ and the α's in $(3.1.2)$ or by transforming the basic relations in Pfeifer, 2004, p. 12.

We claim that the following relation holds for the first five matrices in $(6.1.56)$

$$\sum_\sigma g_{\sigma\sigma} \underline{\underline{\gamma}}^\sigma \underline{\underline{\not p}}_1' \underline{\underline{\gamma}}^\rho \underline{\underline{\not p}}_1 \underline{\underline{\gamma}}^\sigma = -2 \underline{\underline{\not p}}_1 \underline{\underline{\gamma}}^\rho \underline{\underline{\not p}}_1'\,. \tag{6.1.58}$$

First, we point out that as in $(3.4.12)$ the Feynman-dagger-matrix $\underline{\underline{\not p}}$ is written like this using $(6.1.53)$

$$\underline{\underline{\not p}} = \underline{\underline{\gamma}}^0 p^0 - \sum_{i=1}^3 \underline{\underline{\gamma}}^i p^{(i)} = \sum_{\mu=0}^3 g_{\mu\mu} \underline{\underline{\gamma}}^\mu p^{(\mu)}\,. \tag{6.1.59}$$

Moreover, we need some relations with γ-matrices and Feynman-dagger-matrices. With the aid of $(6.1.57)$ we find

$$\sum_{\mu} g_{\mu\mu}\underline{\underline{\gamma}}^{\mu}\underline{\underline{\gamma}}^{\mu} = \sum_{\mu} g_{\mu\mu}\frac{1}{2}\left(\underline{\underline{\gamma}}^{\mu}\underline{\underline{\gamma}}^{\mu} + \underline{\underline{\gamma}}^{\mu}\underline{\underline{\gamma}}^{\mu}\right) = \sum_{\mu} g_{\mu\mu}\frac{1}{2}2g_{\mu\mu}\underline{\underline{1}} = \sum_{\mu}\underline{\underline{1}} = 4\cdot\underline{\underline{1}}. \quad (6.1.60)$$

Using $(6.1.59),(6.1.57)$ and $(6.1.60)$ we write

$$\begin{aligned}\sum_{\mu} g_{\mu\mu}\underline{\underline{\gamma}}^{\mu}\underline{\underline{\not{a}}}\underline{\underline{\gamma}}^{\mu} &= \sum_{\mu,\nu} g_{\mu\mu}\underline{\underline{\gamma}}^{\mu}g_{\nu\nu}a^{(\nu)}\underline{\underline{\gamma}}^{\nu}\underline{\underline{\gamma}}^{\mu} \\ &= \sum_{\mu,\nu} g_{\mu\mu}g_{\nu\nu}a^{(\nu)}\underline{\underline{\gamma}}^{\mu}\left(2g_{\mu\nu}\underline{\underline{1}} - \underline{\underline{\gamma}}^{\mu}\underline{\underline{\gamma}}^{\nu}\right) \\ &= \sum_{\mu} g_{\mu\mu}a^{(\mu)}\underline{\underline{\gamma}}^{\mu}2\cdot\underline{\underline{1}} - \sum_{\mu} g_{\mu\mu}\underline{\underline{\gamma}}^{\mu}\underline{\underline{\gamma}}^{\mu}\underline{\underline{\not{a}}} \\ &= \underline{\underline{\not{a}}}2 - 4\underline{\underline{\not{a}}} = -2\underline{\underline{\not{a}}}.\end{aligned} \quad (6.1.61)$$

Similar manipulations yield

$$\begin{aligned}\underline{\underline{\not{b}\not{a}}} &= \sum_{\mu,\nu} g_{\mu\mu}g_{\nu\nu}\underline{\underline{\gamma}}^{\mu}b^{(\mu)}\underline{\underline{\gamma}}^{\nu}a^{(\nu)} \\ &= \sum_{\mu,\nu} g_{\mu\mu}g_{\nu\nu}\left(2g_{\mu\nu}\underline{\underline{1}} - \underline{\underline{\gamma}}^{\nu}\underline{\underline{\gamma}}^{\mu}\right)a^{(\nu)}b^{(\mu)} \\ &= 2\sum_{\mu} g_{\mu\mu}a^{(\mu)}b^{(\mu)}\underline{\underline{1}} - \underline{\underline{\not{a}\not{b}}} = 2\underline{a}\cdot\underline{b}\underline{\underline{1}} - \underline{\underline{\not{a}\not{b}}}.\end{aligned} \quad (6.1.62)$$

Using $(6.1.61)$ and $(6.1.62)$ we obtain

$$\begin{aligned}\sum_{\mu} g_{\mu\mu}\underline{\underline{\gamma}}^{\mu}\underline{\underline{\not{a}\not{b}}}\underline{\underline{\gamma}}^{\mu} &= \sum_{\mu} g_{\mu\mu}\underline{\underline{\gamma}}^{\mu}\underline{\underline{\not{a}}}\sum_{\lambda} g_{\lambda\lambda}b^{(\lambda)}\underline{\underline{\gamma}}^{\lambda}\underline{\underline{\gamma}}^{\mu} \\ &= \sum_{\mu} g_{\mu\mu}\underline{\underline{\gamma}}^{\mu}\underline{\underline{\not{a}}}\sum_{\lambda} g_{\lambda\lambda}b^{(\lambda)}\left(2g_{\lambda\mu} - \underline{\underline{\gamma}}^{\mu}\underline{\underline{\gamma}}^{\lambda}\right) \\ &= \sum_{\mu} g_{\mu\mu}b^{(\mu)}\underline{\underline{\lambda}}^{\mu}\underline{\underline{\not{a}}}2 - \sum_{\mu} g_{\mu\mu}\underline{\underline{\gamma}}^{\mu}\underline{\underline{\not{a}}}\underline{\underline{\gamma}}^{\mu}\underline{\underline{\not{b}}} \\ &= 2\underline{\underline{\not{b}\not{a}}} + 2\underline{\underline{\not{a}\not{b}}} \\ &= 4\underline{a}\cdot\underline{b}\,\underline{\underline{1}} - 2\underline{\underline{\not{a}\not{b}}} + 2\underline{\underline{\not{a}\not{b}}} = 4\underline{a}\cdot\underline{b}\,\underline{\underline{1}}.\end{aligned} \quad (6.1.63)$$

With $(6.1.62)$ and $(6.1.63)$ we write

$$\begin{aligned}\sum_{\mu} g_{\mu\mu}\underline{\underline{\gamma}}^{\mu}\underline{\underline{\not{a}\not{b}\not{c}}}\underline{\underline{\gamma}}^{\mu} &= \sum_{\mu} g_{\mu\mu}\underline{\underline{\gamma}}^{\mu}\underline{\underline{\not{a}\not{b}}}\sum_{\nu} g_{\nu\nu}c^{(\nu)}\underline{\underline{\gamma}}^{\nu}\underline{\underline{\gamma}}^{\mu} \\ &= \sum_{\mu} g_{\mu\mu}\underline{\underline{\gamma}}^{\mu}\underline{\underline{\not{a}\not{b}}}\sum_{\nu} g_{\nu\nu}c^{(\nu)}\left(2g_{\mu\nu}\underline{\underline{1}} - \underline{\underline{\gamma}}^{\mu}\underline{\underline{\gamma}}^{\nu}\right) \\ &= 2\underline{\underline{\not{c}\not{a}\not{b}}} - \sum_{\mu} g_{\mu\mu}\underline{\underline{\gamma}}^{\mu}\underline{\underline{\not{a}\not{b}}}\underline{\underline{\gamma}}^{\mu}\underline{\underline{\not{c}}} \\ &= 2\underline{\underline{\not{c}\not{a}\not{b}}} - 4\underline{a}\cdot\underline{b}\underline{\underline{\not{c}}} \\ &= 2\underline{\underline{\not{c}}}\left(2\underline{a}\cdot\underline{b}\underline{\underline{1}} - \underline{\underline{\not{b}\not{a}}}\right) - 4\underline{a}\cdot\underline{b}\underline{\underline{\not{c}}} \\ &= 4\underline{\underline{\not{c}}}\underline{a}\cdot\underline{b} - 2\underline{\underline{\not{c}\not{b}\not{a}}} - 4\underline{\underline{\not{c}}}\underline{a}\cdot\underline{b} = -2\underline{\underline{\not{c}\not{b}\not{a}}}.\end{aligned} \quad (6.1.64)$$

From $(6.1.55)$ and $(6.1.57)$ we obtain

$$\begin{aligned}\mathrm{Tr}\left(\underline{\underline{\not a}}\,\underline{\underline{\not b}}\right) &= \mathrm{Tr}\left(\underline{\underline{\not b}}\,\underline{\underline{\not a}}\right) = \frac{1}{2}\mathrm{Tr}\left(\underline{\underline{\not a}}\,\underline{\underline{\not b}} + \underline{\underline{\not b}}\,\underline{\underline{\not a}}\right) \\ &= \frac{1}{2}\mathrm{Tr}\left(\sum_{\mu\nu} g_{\mu\mu} g_{\nu\nu}\left(\underline{\underline{\gamma}}^{\mu}\underline{\underline{\gamma}}^{\nu} + \underline{\underline{\gamma}}^{\nu}\underline{\underline{\gamma}}^{\mu}\right) a^{(\mu)} b^{(\nu)}\right) \\ &= \frac{1}{2}\sum_{\mu\nu} a^{(\mu)} b^{(\nu)} g_{\mu\mu} g_{\nu\nu}\mathrm{Tr}\left(2 g_{\mu\nu}\underline{\underline{1}}\right) \\ &= \sum_{\mu} g_{\mu\mu} a^{(\mu)} b^{(\mu)}\mathrm{Tr}\left(\underline{\underline{1}}\right) = 4\underline{a}\cdot\underline{b}.\end{aligned} \tag{6.1.65}$$

We go on to $\mathrm{Tr}\left(\underline{\underline{\not a}}\,\underline{\underline{\not b}}\,\underline{\underline{\not c}}\,\underline{\underline{\not d}}\right)$ and apply $(6.1.62)$ like this

$$\begin{aligned}\mathrm{Tr}\left(\underline{\underline{\not a}}\,\underline{\underline{\not b}}\,\underline{\underline{\not c}}\,\underline{\underline{\not d}}\right) &= 2\underline{a}\cdot\underline{b}\mathrm{Tr}\left(\underline{\underline{1}}\,\underline{\underline{\not c}}\,\underline{\underline{\not d}}\right) - \mathrm{Tr}\left(\underline{\underline{\not b}}\,\underline{\underline{\not a}}\,\underline{\underline{\not c}}\,\underline{\underline{\not d}}\right) \\ &= 2\underline{a}\cdot\underline{b}\mathrm{Tr}\left(\underline{\underline{\not c}}\,\underline{\underline{\not d}}\right) - 2\underline{a}\cdot\underline{c}\mathrm{Tr}\left(\underline{\underline{\not b}}\,\underline{\underline{\not d}}\right) + \mathrm{Tr}\left(\underline{\underline{\not b}}\,\underline{\underline{\not c}}\,\underline{\underline{\not a}}\,\underline{\underline{\not d}}\right) \\ &= 2\underline{a}\cdot\underline{b}\mathrm{Tr}\left(\underline{\underline{\not c}}\,\underline{\underline{\not d}}\right) - 2\underline{a}\cdot\underline{c}\mathrm{Tr}\left(\underline{\underline{\not b}}\,\underline{\underline{\not d}}\right) + 2\underline{a}\cdot\underline{d}\mathrm{Tr}\left(\underline{\underline{\not b}}\,\underline{\underline{\not c}}\right) - \mathrm{Tr}\left(\underline{\underline{\not b}}\,\underline{\underline{\not c}}\,\underline{\underline{\not d}}\,\underline{\underline{\not a}}\right).\end{aligned} \tag{6.1.66}$$

Due to $(6.1.55)$ and $(6.1.65)$ we get

$2\mathrm{Tr}\left(\underline{\underline{\not a}}\,\underline{\underline{\not b}}\,\underline{\underline{\not c}}\,\underline{\underline{\not d}}\right) = 2\underline{a}\cdot\underline{b}\mathrm{Tr}\left(\underline{\underline{\not c}}\,\underline{\underline{\not d}}\right) - 2\underline{a}\cdot\underline{c}\mathrm{Tr}\left(\underline{\underline{\not b}}\,\underline{\underline{\not d}}\right) + 2\underline{a}\cdot\underline{d}\mathrm{Tr}\left(\underline{\underline{\not b}}\,\underline{\underline{\not c}}\right)$, i.e.

$\mathrm{Tr}\left(\underline{\underline{\not a}}\,\underline{\underline{\not b}}\,\underline{\underline{\not c}}\,\underline{\underline{\not d}}\right) = 4\left(\underline{a}\cdot\underline{b}\right)\left(\underline{c}\cdot\underline{d}\right) - 4\left(\underline{a}\cdot\underline{c}\right)\left(\underline{b}\cdot\underline{d}\right) + 4\left(\underline{a}\cdot\underline{d}\right)\left(\underline{b}\cdot\underline{c}\right).$

Furthermore, due to the structure of Feynman daggers and of the $\underline{\underline{\gamma}}$'s

$$\mathrm{Tr}\,\underline{\underline{\not a}} = 0$$

holds. One can show that the trace of an odd number of Feynman daggers vanishes

$$\mathrm{Tr}\left(\underline{\underline{\not a}}_1\underline{\underline{\not a}}_2\cdots\underline{\underline{\not a}}_{2n+1}\right) = 0 \quad \text{(given without proof)}. \tag{6.1.66a}$$

Now we can conclude the proof of $(6.1.58)$. For this, the central matrix $\underline{\underline{\gamma}}^{\rho}$ can be interpreted as a Feynman-dagger-matrix. In order to transfer $\underline{\underline{\gamma}}^{\rho}$ into the "slash" notation we introduce the four-vector

$$\underline{G}^{(\rho)} \equiv \left(G^{(\rho)(0)}, G^{(\rho)(1)}, G^{(\rho)(2)}, G^{(\rho)(3)}\right) = \left(g_{\rho 0}, g_{\rho 1}, g_{\rho 2}, g_{\rho 3}\right). \tag{6.1.67}$$

Due to $(6.1.58)$ we state

$$\underline{\underline{\not G}}^{(\rho)} = \sum_{\nu=0}^{3} g_{\nu\nu}\underline{\underline{\gamma}}^{\nu} G^{(\rho)(\nu)} = \sum_{\nu=0}^{3} g_{\nu\nu}\underline{\underline{\gamma}}^{\nu} g_{\rho\nu} = \underline{\underline{\gamma}}^{\rho}. \tag{6.1.68}$$

We see that $\underline{\underline{\gamma}}^{\rho}$ is a Feynman-dagger-matrix. Therefore, we can apply $(6.1.64)$ to the left hand side of $(6.1.58)$ producing

$$\sum_{\sigma} g_{\sigma\sigma}\underline{\underline{\gamma}}^{\sigma}\,\underline{\underline{\not p}}_1'\underline{\underline{\gamma}}^{\rho}\,\underline{\underline{\not p}}_1\underline{\underline{\gamma}}^{\sigma} = -2\underline{\underline{\not p}}_1\underline{\underline{\gamma}}^{\rho}\,\underline{\underline{\not p}}_1', \tag{6.1.69}$$

as claimed in $(6.1.58)$ which we insert in $(6.1.56)$ using $(6.1.63)$ and $(6.1.65)$

$$\bar{T}_{2,\text{ur}} \cdot \left(\frac{2\varepsilon_0 c}{\hbar}\right)^2 (2mc)^4 \frac{\left(\underline{p}-\underline{p}_1'\right)^2 \left(\underline{p}-\underline{p}_2'\right)^2}{\hbar^4}$$

$$= \mathrm{Tr}\sum_{\lambda\mu} g_{\lambda\lambda} g_{\mu\mu} \underline{\underline{\gamma}}^{\lambda} \underline{\underline{\not{p}}}_1' \underline{\underline{\gamma}}^{\mu} \underline{\underline{\not{p}}}_1 \underline{\underline{\gamma}}^{\lambda} \underline{\underline{\not{p}}}_2' \underline{\underline{\gamma}}^{\mu} \underline{\underline{\not{p}}}_2 = -2\mathrm{Tr}\left(\sum_{\mu} g_{\mu\mu} \underline{\underline{\not{p}}}_1 \underline{\underline{\gamma}}^{\mu} \underline{\underline{\not{p}}}_1' \underline{\underline{\not{p}}}_2' \underline{\underline{\gamma}}^{\mu} \underline{\underline{\not{p}}}_2\right) \quad (6.1.70)$$

$$= -2\mathrm{Tr}\left(\underline{\underline{\not{p}}}_1 4\underline{p}_1' \cdot \underline{p}_2' \underline{\underline{\not{p}}}_2\right) = -8\underline{p}_1' \cdot \underline{p}_2' \mathrm{Tr}\left(\underline{\underline{\not{p}}}_1 \underline{\underline{\not{p}}}_2\right)$$

$$= -32\left(\underline{p}_1' \cdot \underline{p}_2'\right)\left(\underline{p}_1 \cdot \underline{p}_2\right).$$

Now, we write $\bar{T}_1$, $(6.1.50)$, in the ultra relativistic limit using $(6.1.55)$

$$\bar{T}_{1.\text{ur}} \left(\frac{2\varepsilon_0 c}{\hbar}\right)^2 (2mc)^4 \left(\frac{\underline{p}_1 - \underline{p}_1'}{\hbar}\right)^4$$

$$= \sum_{\mu\lambda} g_{\mu\mu} g_{\nu\nu} \mathrm{Tr}\left(\underline{\underline{\gamma}}^{\mu} \underline{\underline{\not{p}}}_1 \underline{\underline{\gamma}}^{\lambda} \underline{\underline{\not{p}}}_1'\right) \mathrm{Tr}\left(\underline{\underline{\gamma}}^{\mu} \underline{\underline{\not{p}}}_2 \underline{\underline{\gamma}}^{\lambda} \underline{\underline{\not{p}}}_2'\right) \quad (6.1.71)$$

$$= \sum_{\mu\lambda} g_{\mu\mu} g_{\nu\nu} \mathrm{Tr}\left(\underline{\underline{\not{p}}}_1' \underline{\underline{\gamma}}^{\mu} \underline{\underline{\not{p}}}_1 \underline{\underline{\gamma}}^{\lambda}\right) \mathrm{Tr}\left(\underline{\underline{\not{p}}}_2' \underline{\underline{\gamma}}^{\mu} \underline{\underline{\not{p}}}_2 \underline{\underline{\gamma}}^{\lambda}\right)$$

We rewrite the first trace in $(6.1.71)$ interpreting the $\underline{\underline{\gamma}}$'s as Feynman-dagger-matrices applying $(6.1.66)$ and $(6.1.68)$

$$\mathrm{Tr}\left(\underline{\underline{\not{p}}}_1' \underline{\underline{\gamma}}^{\mu} \underline{\underline{\not{p}}}_1 \underline{\underline{\gamma}}^{\lambda}\right) = \mathrm{Tr}\left(\underline{\underline{\not{p}}}_1' \underline{\underline{\not{G}}}^{(\mu)} \underline{\underline{\not{p}}}_1 \underline{\underline{\not{G}}}^{(\lambda)}\right)$$

$$= 4\left(\left(\underline{p}_1' \cdot \underline{G}^{(\mu)}\right)\left(\underline{p}_1 \cdot \underline{G}^{(\lambda)}\right) + \left(\underline{p}_1' \cdot \underline{G}^{(\lambda)}\right)\left(\underline{p}_1 \cdot \underline{G}^{(\mu)}\right) - \left(\underline{p}_1' \cdot \underline{p}_1\right)\left(\underline{G}^{(\mu)} \cdot \underline{G}^{(\lambda)}\right)\right)$$

$$= 4(\left(\sum_{\rho} p_1'^{(\rho)} g_{\rho\rho} g_{\mu\rho}\right)\left(\sum_{\sigma} p_1^{(\sigma)} g_{\sigma\sigma} g_{\lambda\sigma}\right)$$

$$+\left(\sum_{\rho} p_1'^{(\rho)} g_{\rho\rho} g_{\lambda\rho}\right)\left(\sum_{\sigma} p_1^{(\sigma)} g_{\sigma\sigma} g_{\mu\sigma}\right) \quad (6.1.72)$$

$$-\left(\underline{p}_1 \cdot \underline{p}_1'\right)\left(\sum_{\sigma} g_{\sigma\sigma} g_{\mu\sigma} g_{\lambda\sigma}\right))$$

$$= 4\left(p_1'^{(\mu)} p_1^{(\lambda)} + p_1'^{(\lambda)} p_1^{(\mu)} - \left(\underline{p}_1 \cdot \underline{p}_1'\right) g_{\mu\lambda}\right).$$

We now work on $(6.1.71)$ and insert $(6.1.72)$

$$\bar{T}_{1,\mathrm{ur}}\left(\frac{\varepsilon_0 c}{\hbar}\right)^2 (2mc)^4\left(\frac{\underline{p}_1-\underline{p}_1'}{\hbar}\right)^4$$

$$=\sum_{\mu\lambda}\mathrm{Tr}\left(\underline{\underline{\not{p}_1'}}\,\underline{\underline{\gamma}}^{\mu}\,\underline{\underline{\not{p}_1}}\,\underline{\underline{\gamma}}^{\lambda}\right)\cdot g_{\mu\mu}g_{\lambda\lambda}\mathrm{Tr}\left(\underline{\underline{\not{p}_2'}}\,\underline{\underline{\gamma}}^{\mu}\,\underline{\underline{\not{p}_2}}\,\underline{\underline{\gamma}}^{\lambda}\right)$$

$$=\sum_{\mu\lambda}4\left(p_1'^{(\mu)}p_1^{(\lambda)}+p_1'^{(\lambda)}p_1^{(\mu)}-\left(\underline{p}_1'\cdot\underline{p}_1\right)g_{\mu\lambda}\right)$$

$$\cdot 4\left(p_2'^{(\mu)}g_{\mu\mu}p_2^{(\lambda)}g_{\lambda\lambda}+p_2'^{(\lambda)}g_{\lambda\lambda}p_2^{(\mu)}g_{\mu\mu}-\left(\underline{p}_2'\cdot\underline{p}_2\right)g_{\mu\lambda}\right)$$

and taking into account $\left(g_{\tau\tau}\right)^2=1$

$$=16\,[\left(\sum_{\mu}p_1'^{(\mu)}p_2'^{(\mu)}g_{\mu\mu}\right)\left(\sum_{\lambda}p_1^{(\lambda)}p_2^{(\lambda)}g_{\lambda\lambda}\right)$$

$$+\left(\sum_{\mu}p_1'^{(\mu)}p_2^{(\mu)}g_{\mu\mu}\right)\left(\sum_{\lambda}p_1^{(\lambda)}p_2'^{(\lambda)}g_{\lambda\lambda}\right)$$

$$-\sum_{\mu}p_1'^{(\mu)}p_1^{(\mu)}g_{\mu\mu}\left(\underline{p}_2'\cdot\underline{p}_2\right)$$

$$+\left(\sum_{\lambda}p_1'^{(\lambda)}p_2^{(\lambda)}g_{\lambda\lambda}\right)\left(\sum_{\mu}p_1^{(\mu)}p_2'^{(\mu)}g_{\mu\mu}\right)$$

$$+\left(\sum_{\lambda}p_1'^{(\lambda)}p_2'^{(\lambda)}g_{\lambda\lambda}\right)\left(\sum_{\mu}p_1^{(\mu)}p_2^{(\mu)}g_{\mu\mu}\right)$$

$$-\sum_{\mu}p_1'^{(\mu)}p_1^{(\mu)}g_{\mu\mu}\left(\underline{p}_2'\cdot\underline{p}_2\right)$$

$$-\left(\sum_{\mu}p_2'^{(\mu)}p_2^{(\mu)}g_{\mu\mu}\right)\left(\underline{p}_1'\cdot\underline{p}_1\right)-\left(\sum_{\mu}p_2'^{(\mu)}p_2^{(\mu)}g_{\mu\mu}\right)\left(\underline{p}_1'\cdot\underline{p}_1\right)\qquad(6.1.73)$$

$$+\left(\underline{p}_1'\cdot\underline{p}_1\right)\left(\underline{p}_2'\cdot\underline{p}_2\right)\sum_{\mu}g_{\mu\mu}^2]$$

$$=16\,[2\left(\underline{p}_1'\cdot\underline{p}_2'\right)\left(\underline{p}_1\cdot\underline{p}_2\right)+2\left(\underline{p}_1'\cdot\underline{p}_2\right)\left(\underline{p}_1\cdot\underline{p}_2'\right)$$

$$-4\left(\underline{p}_1'\cdot\underline{p}_1\right)\left(\underline{p}_2'\cdot\underline{p}_2\right)+4\left(\underline{p}_1'\cdot\underline{p}_1\right)\left(\underline{p}_2'\cdot\underline{p}_2\right)]$$

$$=32\,[\left(\underline{p}_1'\cdot\underline{p}_2'\right)\left(\underline{p}_1\cdot\underline{p}_2\right)+\left(\underline{p}_1'\cdot\underline{p}_2\right)\left(\underline{p}_1\cdot\underline{p}_2'\right)].$$

We put our results together. We rewrite the unpolarized, squared invariant matrix element $(6.1.52)$ in the ultra relativistic limit using $(6.1.73)$ and $(6.1.70)$ like this

$$\overline{\left|M_{fi}\left(\underline{p}_1',r_1',\underline{p}_2',r_2',\underline{p}_1,r_1,\underline{p}_2,r_2\right)\right|}_{\text{ur}}^2 = \overline{T}_{1,\text{ur}} - \overline{T}_{2,\text{ur}} + \left(\underline{p}_1' \rightleftarrows \underline{p}_2'\right)$$

$$= \left(\frac{\hbar}{2\varepsilon_0 c}\right)^2 32\left(\frac{\hbar}{2mc}\right)^4 [\frac{\left(\underline{p}_1'\cdot\underline{p}_2'\right)\left(\underline{p}_1\cdot\underline{p}_2\right)+\left(\underline{p}_1'\cdot\underline{p}_2\right)\left(\underline{p}_1\cdot\underline{p}_2'\right)}{\left(\underline{p}_1-\underline{p}_1'\right)^4} \tag{6.1.74}$$

$$-\frac{(-1)\left(\underline{p}_1'\cdot\underline{p}_2'\right)\left(\underline{p}_1\cdot\underline{p}_2\right)}{\left(\underline{p}_1-\underline{p}_1'\right)^2\left(\underline{p}_1-\underline{p}_2'\right)^2}+\frac{\left(\underline{p}_1'\cdot\underline{p}_2'\right)\left(\underline{p}_1\cdot\underline{p}_2\right)+\left(\underline{p}_2'\cdot\underline{p}_2\right)\left(\underline{p}_1\cdot\underline{p}_1'\right)}{\left(\underline{p}_1-\underline{p}_2'\right)^4}$$

$$-\frac{(-1)\left(\underline{p}_1'\cdot\underline{p}_2'\right)\left(\underline{p}_1\cdot\underline{p}_2\right)}{\left(\underline{p}_1-\underline{p}_1'\right)^2\left(\underline{p}_1-\underline{p}_2'\right)^2}].$$

The second and the last term in the square brackets in $(6.1.74)$ agree. We insert $(6.1.74)$ in $(6.1.44)$ and obtain the differential cross section in the ultra relativistic limit with $v_1 \simeq c$ and $\left|\vec{p}_1'\right|c \simeq E$

$$\overline{\left(\frac{d\sigma}{d\Omega_1'}\right)_{\text{ur}}} = \frac{\left(emc^2\right)^4}{\left(2\pi E\right)^2 4c}\frac{\left|\vec{p}_1'\right|}{E\hbar^6}\left(\frac{\mathcal{E}'}{\mathcal{E}}\right)^8 \overline{\left|M_{fi}\right|}_{\text{ur}}^2 \Big|_{\substack{\underline{p}_2'=\underline{p}_1+\underline{p}_2-\underline{p}_1' \\ E'=E}}$$

$$= \frac{e^4}{\varepsilon_0^2\left(2\pi\right)^2 E^2 8}\left(\frac{\mathcal{E}'}{\mathcal{E}}\right)^8 [\frac{\left(\underline{p}_1'\cdot\underline{p}_2'\right)\left(\underline{p}_1\cdot\underline{p}_2\right)+\left(\underline{p}_1'\cdot\underline{p}_2\right)\left(\underline{p}_1\cdot\underline{p}_2'\right)}{\left(\underline{p}_1-\underline{p}_1'\right)^4} \tag{6.1.75}$$

$$+\frac{\left(\underline{p}_1'\cdot\underline{p}_2'\right)\left(\underline{p}_1\cdot\underline{p}_2\right)+\left(\underline{p}_2'\cdot\underline{p}_2\right)\left(\underline{p}_1\cdot\underline{p}_1'\right)}{\left(\underline{p}_1-\underline{p}_2'\right)^4}$$

$$+2\frac{\left(\underline{p}_1'\cdot\underline{p}_2'\right)\left(\underline{p}_1\cdot\underline{p}_2\right)}{\left(\underline{p}_1-\underline{p}_1'\right)^2\left(\underline{p}_1-\underline{p}_2'\right)^2}]\Big|_{\substack{\underline{p}_2'=\underline{p}_1+\underline{p}_2-\underline{p}_1' \\ E'=E}} .$$

The dimension of this cross section is [area] as given at the beginning of this chapter. We evaluate the result $(6.1.75)$ in terms of the scattering angle Θ i.e. of the angle between $\vec{p}_1$ and $\vec{p}_1'$. In the centre-of-mass frame we have due to $(6.1.21)$

$$\underline{p}_1 = \left(\frac{E_1}{c},\vec{p}_1\right) \equiv \left(\frac{E}{c},\vec{p}\right), \qquad \underline{p}_2 = \left(\frac{E_2}{c},\vec{p}_2\right) \equiv \left(\frac{E}{c},-\vec{p}\right),$$

$$\underline{p}_1' = \left(\frac{E_1'}{c},\vec{p}_1'\right) \equiv \left(\frac{E'}{c},\vec{p}'\right), \qquad \underline{p}_2' = \left(\frac{E_2'}{c},\vec{p}_2'\right) \equiv \left(\frac{E'}{c},-\vec{p}'\right), \tag{6.1.76}$$

$$\text{and } \Theta = \sphericalangle\left(\vec{p}',\vec{p}\right).$$

In the ultra relativistic limit $E \gg mc^2$ the relations $\vec{p}^2c^2 = E^2 - m^2c^4 \simeq E^2$ and $\underline{p}'^2c^2 \simeq E'^2$ hold. The scalar products needed in $(6.1.75)$ are (c.f. $(3.4.35a)$ and $(6.1.80)$)

$$
\begin{aligned}
&\underline{p}_1^2 = \frac{E^2}{c^2} - \vec{p}_1^2 = m^2c^2 = \underline{p}_1'^2 = \underline{p}_2^2 = \underline{p}_2'^2 \\
&\underline{p}_1 \cdot \underline{p}_1' = \frac{EE'}{c^2} - \vec{p}_1 \cdot \vec{p}_1' = \frac{EE'}{c^2} - |\vec{p}||\vec{p}'|\cos\Theta \simeq 2\frac{EE'}{c^2}\sin^2\frac{\Theta}{2}, \\
&\underline{p}_1 \cdot \underline{p}_2 = \frac{E^2}{c^2} - \vec{p}_1 \cdot \vec{p}_2 = \frac{E^2}{c^2} - \vec{p}(-\vec{p}) \simeq 2\frac{E^2}{c^2}, \\
&\underline{p}_1 \cdot \underline{p}_2' = \frac{EE'}{c^2} - \vec{p}_1 \cdot \vec{p}_2' = \frac{EE'}{c^2} + |\vec{p}||\vec{p}'|\cos\Theta \simeq 2\frac{EE'}{c^2}\cos^2\frac{\Theta}{2}, \\
&\underline{p}_1' \cdot \underline{p}_2 = \frac{EE'}{c^2} - \vec{p}_1' \cdot \vec{p}_2 = \frac{EE'}{c^2} + \vec{p} \cdot \vec{p}' \simeq 2\frac{EE'}{c^2}\cos^2\frac{\Theta}{2}, \\
&\underline{p}_1' \cdot \underline{p}_2' = \frac{E'^2}{c^2} - \vec{p}_1' \cdot \vec{p}_2' = \frac{E'^2}{c^2} + \vec{p}'^2 \simeq 2\frac{E'^2}{c^2}, \\
&\underline{p}_2 \cdot \underline{p}_2' = \frac{EE'}{c^2} - \vec{p} \cdot \vec{p}' \simeq \frac{EE'}{c^2}\sin^2\frac{\Theta}{2}.
\end{aligned}
\qquad (6.1.77)
$$

Due to $E = E'$ and $(6.1.76)$ the denominators in $(6.1.75)$ are given by

$$
\begin{aligned}
\left(\underline{p}_1 - \underline{p}_1'\right)^2 &= \underline{p}_1^2 + \underline{p}_1'^2 - 2\underline{p}_1 \cdot \underline{p}_1' \simeq 2m^2c^2 - \frac{4E^2}{c^2}\sin^2\frac{\Theta}{2} \\
&\simeq -\frac{4E^2}{c^2}\sin^2\frac{\Theta}{2}, \\
\left(\underline{p}_1 - \underline{p}_2'\right)^2 &= \underline{p}_1^2 + \underline{p}_2'^2 - 2\underline{p}_1 \cdot \underline{p}_2' \simeq 2m^2c^2 - \frac{4E^2}{c^2}\cos^2\frac{\Theta}{2} \\
&\simeq -\frac{4E^2}{c^2}\cos^2\frac{\Theta}{2}.
\end{aligned}
\qquad (6.1.78)
$$

Inserting $(6.1.77)$ and $(6.1.78)$ in $(6.1.75)$ yields

$$
\begin{aligned}
\overline{\left(\frac{d\sigma}{d\Omega_1'}\right)_{\text{ur}}} &\simeq \frac{e^4}{\varepsilon_0{}^2(2\pi)^2E^2 8}\left(\frac{\mathcal{E}'}{\mathcal{E}}\right)^8 [\frac{2E^2 2E^2 + \left(2E^2\cos^2\frac{\Theta}{2}\right)^2}{\left(4E^2\sin^2\frac{\Theta}{2}\right)^2} \\
&+ \frac{2E^2 2E^2 + \left(2E^2\sin^2\frac{\Theta}{2}\right)^2}{\left(4E^2\cos^2\frac{\Theta}{2}\right)^2} + 2\frac{2E^2 2E^2}{4E^2\sin^2\frac{\Theta}{2}4E^2\cos^2\frac{\Theta}{2}}] \\
&\simeq \frac{e^4}{\varepsilon_0{}^2(2\pi)^2E^2 8\cdot 4}\left(\frac{\mathcal{E}'}{\mathcal{E}}\right)^8 [\frac{1+\cos^4\frac{\Theta}{2}}{\sin^4\frac{\Theta}{2}} + \frac{1+\sin^4\frac{\Theta}{2}}{\cos^4\frac{\Theta}{2}} \\
&+ \frac{2}{\sin^2\frac{\Theta}{2}\cos^2\frac{\Theta}{2}}].
\end{aligned}
\qquad (6.1.79)
$$

With the aid of the trigonometric relations

$$\sin^2\frac{\Theta}{2}=\frac{1-\cos\Theta}{2},\ \cos^2\frac{\Theta}{2}=\frac{1+\cos\Theta}{2},\ \sin\frac{\Theta}{2}\cos\frac{\Theta}{2}=\frac{1}{2}\sin\Theta \qquad (6.1.80)$$

we reduce $(6.1.79)$ and we put the dimensionless factor $\left(\frac{\mathcal{E}'}{\mathcal{E}}\right)^8$ equal to 1

$$\overline{\left(\frac{d\sigma}{d\Omega_1'}\right)_{\text{ur}}}\frac{(2\pi)^2E^2 8\cdot 4\,\varepsilon_0^2}{e^4}$$
$$=[\frac{\left(1+\cos^4\frac{\Theta}{2}\right)\cos^4\frac{\Theta}{2}+\left(1+\sin^4\frac{\Theta}{2}\right)\sin^4\frac{\Theta}{2}}{\sin^4\frac{\Theta}{2}\cos^4\frac{\Theta}{2}}+\frac{2}{\sin^2\frac{\Theta}{2}\cos^2\frac{\Theta}{2}}]$$
$$=\frac{8}{\sin^4\Theta}[2\left(1+\left(\frac{1+\cos\Theta}{2}\right)^2\right)\left(\frac{1+\cos\Theta}{2}\right)^2$$
$$+2\left(1+\left(\frac{1-\cos\Theta}{2}\right)^2\right)\left(\frac{1-\cos\Theta}{2}\right)^2+\sin^2\Theta]$$
$$=\frac{1}{\sin^4\Theta}[8\left(1+\cos^2\Theta+\sin^2\Theta\right)+(1+\cos\Theta)^4+(1-\cos\Theta)^4\,]$$
$$=\frac{18+12\cos^2\Theta+2\cos^4\Theta}{\sin^4\Theta}=2\frac{\left(3+\cos^2\Theta\right)^2}{\sin^4\Theta}. \qquad (6.1.81)$$

With the help of the dimensionless fine-structure constant

$$\alpha=\frac{e^2}{4\pi\varepsilon_0\hbar c}\simeq\frac{1}{137}\quad \text{(in the system of units SI)} \qquad (6.1.82)$$

the differential scattering cross section of unpolarized electrons in the ultra relativistic limit reads

$$\overline{\left(\frac{d\sigma}{d\Omega'}\right)_{\text{ur}}}=\frac{\alpha^2\hbar^2c^2}{4E^2}\frac{\left(3+\cos^2\Theta\right)^2}{\sin^4\Theta}, \qquad (6.1.83)$$

which agrees with the result of quantum electrodynamics. But in our derivation we have used quantum field theory, which is better backed up. As is well known the result of quantum electrodynamics – and also our result – agrees very well with measurements.

6.2 The cross section for the scattering of photons by free electrons

This so-called Compton scattering is treated in section 5.7. Due to $(5.7.10)$ the scattering matrix element reads

$$S_{fi} \simeq -\mathrm{i}\frac{e^2}{V^2}\frac{mc^2}{\sqrt{E(\vec{p})E(\vec{p}\,')2\omega_k 2\omega_{k'}}}(2\pi)^4\hbar^5\delta^4\left(\underline{k}'+\underline{p}'-\underline{k}-\underline{p}\right)$$
$$\cdot\frac{1}{\varepsilon_0\hbar}\left(\frac{\mathcal{A}'\mathcal{E}'}{\mathcal{A}\mathcal{E}}\right)^2\sum_{\mu,\nu=0}^{3}M_{fi,\mu\nu}\left(\underline{k}',\lambda',\underline{p}',r',\underline{k},\lambda,\underline{p},r\right) \qquad \text{with} \tag{6.2.1}$$

$$M_{fi,\mu\nu} = \underline{w}_{r'}^{\dagger}\left(\underline{p}'\right)\underline{\underline{\alpha}}_{\mu}g_{\mu\mu}\underline{\underline{S}}_F\left(\frac{\underline{p}+\underline{k}}{\hbar}\right)\underline{\underline{\beta\alpha}}_{\nu}g_{\nu\nu}\underline{w}_r\left(\underline{p}\right)\varepsilon_{\lambda'}^{\mu}\left(\underline{k}'\right)\varepsilon_{\lambda}^{\nu}\left(\underline{k}\right)$$
$$+\underline{w}_{r'}^{\dagger}\left(\underline{p}'\right)\underline{\underline{\alpha}}_{\mu}g_{\mu\mu}\underline{\underline{S}}_F\left(\frac{\underline{p}+\underline{k}'}{\hbar}\right)\underline{\underline{\beta\alpha}}_{\nu}g_{\nu\nu}\underline{w}_r\left(\underline{p}\right)\varepsilon_{\lambda}^{\mu}\left(\underline{k}\right)\varepsilon_{\lambda'}^{\nu}\left(\underline{k}'\right) \tag{6.2.2}$$

$$\text{and} \qquad \underline{\underline{S}}_F\left(\underline{q}\right) = \frac{\underline{\underline{\not{q}}} + \underline{\underline{1}}mc/\hbar}{\underline{q}^2 - (mc/\hbar)^2 + \mathrm{i}\varepsilon}\,. \tag{6.2.3}$$

In analogy to $(6.1.6)$ the number of states which are reached in the momentum areas d^3k' and d^3p' near the momenta $\vec{k}'$ and $\vec{p}\,'$ starting from the initial state $\left(\vec{k},\vec{p}\right)$ owing to the electromagnetic interaction, reads

$$\left|S_{fi}\left(\underline{k}',\underline{p}',\underline{k},\underline{p}\right)\right|^2 dN = \left|S_{fi}\left(\underline{k}',\underline{p}',\underline{k},\underline{p}\right)\right|^2 V\frac{d^3k'}{(2\pi\hbar)^3}V\frac{d^3p'}{(2\pi\hbar)^3}\,. \tag{6.2.4}$$

As in $(6.1.29)$ we divide this quantity by T, which represents the time period for the scattering process (c.f. $(6.1.16)$) and obtain the corresponding number of reached states per time unit

$$\left|S_{fi}\left(\underline{k}',\underline{p}',\underline{k},\underline{p}\right)\right|^2 V\frac{d^3k'}{(2\pi\hbar)^3}V\frac{d^3p'}{(2\pi\hbar)^3}\frac{1}{T}\,. \tag{6.2.5}$$

The differential cross section which corresponds to $d\tilde{\sigma}$ in $(6.1.30)$ reads

$$d\tilde{\sigma} = \left|S_{fi}\left(\underline{k}',\underline{p}',\underline{k},\underline{p}\right)\right|^2 V\frac{d^3k'}{(2\pi\hbar)^3}V\frac{d^3p'}{(2\pi\hbar)^3}\frac{1}{T\left|v_{\mathrm{rel}}\right|/V}\,. \tag{6.2.6}$$

The expression $\left|v_{rel}\right|/V$ in the denominator represents the incoming photon current analogously to $(6.1.27)$. We treat the square of S_{fi}, $(6.2.1)$. The square of $\delta^4\left(\underline{k}'+\underline{p}'-\underline{k}-\underline{p}\right)$ is replaced using $(6.1.17)$ as follows

$$\left|S_{fi}\left(\underline{k}',\lambda',\underline{p}',r',\underline{k},\lambda,\underline{p},r,\right)\right|^2 = \frac{\left(mc^2\right)^2}{V^3E(\vec{p}\,')E(\vec{p})2\omega_{k'}2\omega_k}(2\pi)^4\hbar^6cT$$
$$\delta^4\left(\underline{k}'+\underline{p}'-\underline{k}-\underline{p}\right)\left(\frac{\mathcal{A}'\mathcal{E}'}{\mathcal{A}\mathcal{E}}\right)^4\frac{e^4}{\varepsilon_0^2\hbar^2}\left|\sum_{\mu,\nu=0}^{3}M_{fi}\left(\underline{k}',\lambda',\underline{p}',r',\underline{k},\lambda,\underline{p},r,\right)\right|^2. \tag{6.2.7}$$

This quantity is dimensionless as we expect. In the following we want to evaluate it in the laboratory frame, where the electron is at rest initially and therefore

$$\underline{p} = (mc, 0, 0, 0), \quad E(\vec{p}) = mc^2 \text{ and } v_{\text{rel}} = c. \tag{6.2.8}$$

Corresponding to $(6.1.30)$ we write

$$d^3k' = d\left|\vec{k}'\right|\left|\vec{k}'\right| d\Omega_{k'}. \tag{6.2.9}$$

We insert $(6.2.7)$ up to $(6.2.9)$ in $(6.2.6)$ and integrate over $\left|\vec{k}'\right|$ and $\vec{p}'$ because we admit all momenta $\vec{p}'$ and all magnitudes $\left|\vec{k}'\right|$ of $\vec{k}'$ analogously to $(6.1.31)$. We obtain

$$\begin{aligned} d\sigma = d\Omega_{k'} \frac{m^2c^4}{(2\pi)^2\, 2\omega_k E(\vec{p})} \frac{e^4}{\varepsilon_0^2\hbar^2} \int \frac{d\left|\vec{k}'\right|\left|\vec{k}'\right|^2}{2\omega_{k'}} \int \frac{d^3p'}{E(\vec{p}')} \delta^4\left(\underline{k}' + \underline{p}' - \underline{k} - \underline{p}\right) \\ \cdot \left(\frac{\mathcal{A}'\mathcal{E}'}{\mathcal{A}\mathcal{E}}\right)^4 \left|\sum_{\mu,\nu} M_{fi,\mu\nu}\left(\underline{k}', \lambda', \underline{p}', r', \underline{k}, \lambda, \underline{p}, r\right)\right|^2, \end{aligned} \tag{6.2.10}$$

which has the dimension [area] as we expect. Due to $(6.1.34)$ we can write $1/E(\vec{p}')$ as an integral like this

$$\begin{aligned} \frac{1}{2E(\vec{p}')} &= c\int_{-\infty}^{\infty} dp'^0 \delta\left(\underline{p}'^2c^2 - m^2c^4\right)\Theta\left(p'^0\right) \\ \text{or } \int \frac{d^3p'}{2E(\vec{p}')} &= c\int_{-\infty}^{\infty} d^4p'\, \delta\left(\underline{p}'^2c^2 - m^2c^4\right)\Theta\left(p'^0\right). \end{aligned} \tag{6.2.11}$$

Analogously to $(6.1.35)$ we pick out the k'- and p'-integrals of $(6.2.10)$ and insert the relation $(4.4.3)$

$$ck'^0 = c\left|\vec{k}'\right| = \omega_{k'} \quad \text{like this} \tag{6.2.12}$$

$$\begin{aligned} I &= \int \frac{d\left|\vec{k}'\right|\left|\vec{k}'\right|^2}{\left|\vec{k}'\right| \cdot c} \int \frac{d^3p'}{2E(\vec{p}')} \delta^4\left(\underline{k}' + \underline{p}' - \underline{k} - \underline{p}\right) \cdot f\left(\underline{k}', \underline{p}', \cdots\right) \text{ with} \\ f\left(\underline{k}', \underline{p}', \cdots\right) &= \left(\frac{\mathcal{A}'\mathcal{E}'}{\mathcal{A}\mathcal{E}}\right)^4 \left|\sum_{\mu,\nu=0}^{3} M_{fi,\mu\nu}\left(\underline{k}', \lambda', \underline{p}', r', \underline{k}, \lambda, \underline{p}, r\right)\right|^2. \end{aligned} \tag{6.2.13}$$

Using $\underline{p}' = \left(\frac{E(\vec{p}')}{c}, \vec{p}'\right)$ and $(6.2.11)$ we obtain

$$I = \int d\left|\vec{k}'\right| \left|\vec{k}'\right| \int d^3 p' \int_{-\infty}^{\infty} d^4 p' \delta\left(\underline{p}'^2 c^2 - m^2 c^4\right) \Theta\left(p'^0\right) \cdot \delta^4\left(\underline{k}' + \underline{p}' - \underline{k} - \underline{p}\right) f\left(\underline{k}', \underline{p}', \cdots\right). \tag{6.2.14}$$

Fourfold application of $(6.1.33)$ to the $\vec{p}'$-integration yields

$$I = \int \frac{d\omega_{k'}\ \omega_{k'}}{c^2} \delta\left(\left(\underline{p} + \underline{k} - \underline{k}'\right)^2 c^2 - m^2 c^4\right) \cdot \Theta\left(p^0 + k^0 - k'^0\right) f\left(\underline{k}', \underline{p}' = \underline{p} + \underline{k} - \underline{k}', \cdots\right) \tag{6.2.15}$$

where we can write using $(6.2.8)$ and $(6.2.12)$

$$\begin{aligned} &\left(\underline{p} + \underline{k} - \underline{k}'\right)^2 c^2 - m^2 c^4 \\ &= m^2 c^4 + 2mc^3\left(k^0 - k'^0\right) + \left(k^0 - k'^0\right)^2 c^2 - \left(\vec{k} - \vec{k}'\right)^2 c^2 - m^2 c^4 \\ &= 2mc^2\left(\omega_k - \omega_{k'}\right) + \left(\left(k^0\right)^2 + \left(k'^0\right)^2 - 2k^0 k'^0 - \vec{k}^2 - \vec{k}'^2 + 2\vec{k} \cdot \vec{k}'\right) c^2 \\ &= 2mc^2\left(\omega_k - \omega_{k'}\right) - 2\omega_k \omega_{k'} + 2\omega_k \omega_{k'} + \cos\Theta \end{aligned} \tag{6.2.16}$$

with the scattering angle Θ (angle between $\vec{k}$ and $\vec{k}'$).

Finally we calculate the $\omega_{k'}$-integral in I, $(6.2.15)$, again using $(6.1.33)$ and $(6.2.16)$

$$\begin{aligned} I &= \int_0^{\omega_{k'}/c+mc} \frac{d\omega_{k'}\omega_{k'}}{c^2} \delta\left(2mc^2\left(\omega_k - \omega_{k'}\right) - 2\omega_k \omega_{k'}\left(1 - \cos\Theta\right)\right) \\ &\quad \cdot f\left(\underline{k}', \underline{p}' = \underline{p} + \underline{k} - \underline{k}', \cdots\right) \\ &= \frac{\omega_{k'}}{c^2} \frac{1}{2\left|mc^2 + \omega_k\left(1 - \cos\Theta\right)\right|} f\left(\underline{k}', \underline{p}', \cdots\right) \Big|_{\omega_{k'} = mc^2\omega_k/\left(mc^2 + \omega_k(1-\cos\Theta)\right)}^{\underline{p}' = \underline{p} + \underline{k} - \underline{k}'} \\ &= \frac{\omega_{k'}^2}{2\omega_k mc^4} f\left(\underline{k}', \underline{p}', \cdots\right) \Big|_{\omega_{k'} = \omega_k/\left(mc^2 + \omega_k(1-\cos\Theta)/mc^2\right)}^{\underline{p}' = \underline{p} + \underline{k} - \underline{k}'} . \end{aligned} \tag{6.2.17}$$

We look into the restraining condition in $(6.2.17)$,

$$\omega_{k'} = \frac{\omega_k}{1 + \omega_k\left(1 - \cos\Theta\right) / mc^2}, \tag{6.2.18}$$

which follows from the root in the delta function that occurred in the derivation of $(6.2.17)$. We replace the photon energy ω by $h\nu = 2\pi\hbar c / \lambda$, which leads to Comptons formula

$$\lambda' = \lambda + 2\pi \frac{\hbar}{mc}\left(1 - \cos\Theta\right). \tag{6.2.19}$$

The wavelength of the scattered photon is increased by an amount which is proportional to the Compton wavelength $\hbar / mc$.

From $(6.2.8)$, $(6.2.10)$, $(6.2.13)$ and $(6.2.17)$ the differential photon scattering cross section results as

$$\frac{d\sigma}{d\Omega_{k'}} = \frac{e^4\omega_{k'}^2}{(2\pi)^2 4\varepsilon_0^2\hbar^2c^2\omega_k^2}\left(\frac{\mathcal{A}'\mathcal{E}'}{\mathcal{A}\mathcal{E}}\right)^4 \cdot \left|\sum_{\mu,\nu=0}^{3} M_{fi,\mu\nu}\left(\underline{k}',\lambda',\underline{p}',r',\underline{k},\lambda,\underline{p},r\right)\right|^2 \Big|_{\omega_{k'}=mc^3\omega_k/\left(mc^2+\omega_k(1-\cos\Theta)\right)}^{\underline{p}'=\underline{p}+\underline{k}-\underline{k}'} . \tag{6.2.20}$$

In analogy with $(6.1.45)$ we will be interested in the case of unpolarized electrons. However, for the time being, we keep the photon polarizations λ and λ'. Thus, $(6.2.20)$ has to be averaged over the initial spins and summed over the final spins of the electrons

$$\begin{aligned}\overline{\frac{d\sigma}{d\Omega_{k'}}}_{\lambda\lambda'} &= \frac{1}{2}\sum_{r',r=1}^{2}\frac{d\sigma}{d\Omega_{k'}}(r',r,\lambda',\lambda) \\ &= \frac{e^4\omega_{k'}^2}{(4\pi)^2 \varepsilon_0^2\hbar^2c^2\omega_k^2}\left(\frac{\mathcal{A}'\mathcal{E}'}{\mathcal{A}\mathcal{E}}\right)^4 \\ &\cdot\frac{1}{2}\sum_{r'.r=1}^{2}\left|\sum_{\mu,\nu=0}^{3} M_{fi,\mu\nu}\left(\underline{k}',\lambda',\underline{p}',r',\underline{k},\lambda,\underline{p},r\right)\right|^2 \Big|_{\omega_{k'}=mc^3\omega_k/\left(mc^2+\omega_k(1-\cos\Theta)\right)}^{\underline{p}'=\underline{p}+\underline{k}-\underline{k}'} .\end{aligned} \tag{6.2.21}$$

Now we work on the term $\sum_{\mu,\nu=0}^{3} M_{fi,\mu\nu}$. The first term on the right hand side of $(6.2.2)$ contains the matrix

$$\underline{\underline{S}}_F\left(\frac{\underline{p}+\underline{k}}{\hbar}\right) \simeq \hbar\frac{\underline{\underline{\not{p}}}+\underline{\underline{\not{k}}}+\underline{\underline{1}}mc}{\left(\underline{p}+\underline{k}\right)^2-m^2c^2} . \tag{6.2.22}$$

Using $(6.2.8\ )$ and $(6.2.12)$ we remodel the denominator in a similar way as in $(6.2.16)$

$$\begin{aligned}\left(\underline{p}+\underline{k}\right)^2-m^2c^2 &= \underline{p}^2+2\underline{p}\cdot\underline{k}+\underline{k}^2-m^2c^2 \\ &= m^2c^2-0+2\underline{p}\cdot\underline{k}+\left(k^0\right)^2-\vec{k}^2-m^2c^2 \\ &= \quad 0+2\underline{p}\cdot\underline{k}+0 = 2\underline{p}\cdot\underline{k}\end{aligned} \tag{6.2.23}$$

and we obtain

$$\underline{\underline{S}}_F\left(\frac{\underline{p}+\underline{k}}{\hbar}\right) \simeq \hbar\frac{\underline{\underline{\not{p}}}+\underline{\underline{\not{k}}}+\underline{\underline{1}}mc}{2\underline{p}\cdot\underline{k}} . \tag{6.2.24}$$

In the second term in $(6.2.2)$ we interchange the dummy indices μ and ν which yields

$$\begin{aligned}
&\sum_{\mu,\nu=0}^{3} M_{fi,\eta\nu}\left(\underline{k}',\lambda',\underline{p}',r',\underline{k},\lambda,\underline{p},r\right) \\
&\quad = \sum_{\mu,\nu=0}^{3} \varepsilon_{\kappa'}^{\mu}\left(\underline{k}'\right)\varepsilon_{\lambda}^{\nu}\left(\underline{k}\right) \\
&\quad \cdot [\underline{w}_{r'}^{\dagger}\left(\underline{p}'\right)\underline{\underline{\alpha}}_{\mu} g_{\mu\mu}\hbar \frac{\underline{\underline{\not p}} + \underline{\underline{\not k}} + \underline{\underline{1}}mc}{2\underline{p}\cdot\underline{k}} \underline{\underline{\beta}}\,\underline{\underline{\alpha}}_{\nu} g_{\nu\nu}\underline{w}_{r}\left(\underline{p}\right) \\
&\quad + \underline{w}_{r'}^{\dagger}\left(\underline{p}'\right)\underline{\underline{\alpha}}_{\nu} g_{\nu\nu}\hbar \frac{\underline{\underline{\not p}} - \underline{\underline{\not k}}' + \underline{\underline{1}}mc}{-2\underline{p}\cdot\underline{k}'} \underline{\underline{\beta}}\,\underline{\underline{\alpha}}_{\mu} g_{\mu\mu}\underline{w}_{r}\left(\underline{p}\right)].
\end{aligned} \tag{6.2.25}$$

In agreement with $(3.4.12)$ one defines the Feynman dagger

$$\begin{aligned}
\underline{\underline{\not\varepsilon}}_{\lambda}\left(\underline{k}\right) &= \underline{\underline{\beta}}\left(\varepsilon_{\lambda}^{0}\left(\underline{k}\right) - \sum_{i=1}^{3}\underline{\underline{\alpha}}_{i}\varepsilon_{\lambda}^{(i)}\left(\underline{k}\right)\right) \\
&= \underline{\underline{\beta}}\sum_{\mu=0}^{3}\underline{\underline{\alpha}}_{\mu} g_{\mu\mu}\varepsilon_{\lambda}^{(\mu)}\left(\underline{k}\right) \qquad \text{with } \underline{\underline{\alpha}}_{0} = \underline{\underline{1}},
\end{aligned} \tag{6.2.26}$$

and one obtains using $\underline{\underline{\beta}}^{2} = \underline{\underline{1}}$

$$\begin{aligned}
\sum_{\mu,\nu=0}^{3} M_{fi,\mu\nu} &= [\underline{w}_{r'}^{\dagger}\left(\underline{p}'\right)\underline{\underline{\beta}}\,\underline{\underline{\not\varepsilon}}_{\lambda'}\left(\underline{k}'\right)\hbar \frac{\underline{\underline{\not p}} + \underline{\underline{\not k}} + \underline{\underline{1}}mc}{2\underline{p}\cdot\underline{k}} \underline{\underline{\not\varepsilon}}_{\lambda}\left(\underline{k}\right)\underline{w}_{r}\left(\underline{p}\right) \\
&+ \underline{w}_{r'}^{\dagger}\left(\underline{p}'\right)\underline{\underline{\beta}}\,\underline{\underline{\not\varepsilon}}_{\lambda}\left(\underline{k}\right)\hbar \frac{\underline{\underline{\not p}} - \underline{\underline{\not k}}' + \underline{\underline{1}}mc}{-2\underline{p}\cdot\underline{k}'} \underline{\underline{\not\varepsilon}}_{\lambda'}\left(\underline{k}'\right)\underline{w}_{r}\left(\underline{p}\right)].
\end{aligned}$$

We define

$$\begin{aligned}
\underline{\underline{\Gamma}} &= \hbar[\underline{\underline{\not\varepsilon}}_{\lambda'}\left(\underline{k}'\right)\frac{\underline{\underline{\not p}} + \underline{\underline{\not k}} + \underline{\underline{1}}mc}{2\underline{p}\cdot\underline{k}}\underline{\underline{\varepsilon}}_{\lambda}\left(\underline{k}\right) \\
&+ \underline{\underline{\not\varepsilon}}_{\lambda}\left(\underline{k}\right)\frac{\underline{\underline{\not p}} + \underline{\underline{\not k}}' + \underline{\underline{1}}mc}{-2\underline{p}\cdot\underline{k}'}\underline{\underline{\varepsilon}}_{\lambda'}\left(\underline{k}'\right)],
\end{aligned} \tag{6.2.27}$$

with which we write

$$\left|\sum_{\mu,\nu=0}^{3} M_{fi,\mu\nu}\right|^{2} = \left|\underline{w}_{r'}^{\dagger}\left(\underline{p}'\right)\underline{\underline{\beta}}\,\underline{\underline{\Gamma}}\,\underline{w}_{r}\left(\underline{p}\right)\right|^{2}. \tag{6.2.28}$$

In analogy with $(6.1.45)$ the mean value of the square of the invariant amplitude in $(6.2.21)$ reads using $(3.4.9)$

$$
\begin{aligned}
&\sum_{r.r'=1}^{2}\frac{1}{2}\left|\sum_{\mu,\nu=0}^{3} M_{fi,\mu\nu}\left(\underline{k}',\lambda',\underline{p}',r',\underline{k},\lambda,\underline{p},r\right)\right|^{2}\\
&=\sum_{r',r=1}^{2}\frac{1}{2}\left(\underline{w}_{r'}^{\dagger}\left(\underline{p}'\right)\underline{\underline{\beta}}\,\underline{\underline{\Gamma}}\,\underline{w}_{r}\left(\underline{p}\right)\right)\left(\underline{w}_{r'}^{\dagger}\left(\underline{p}'\right)\underline{\underline{\beta}}\,\underline{\underline{\Gamma}}\,\underline{w}_{r}\left(\underline{p}\right)\right)^{\dagger}\\
&=\sum_{r',=1}^{2}\frac{1}{2}\,\underline{\bar{w}}_{r'}\left(\underline{p}'\right)\underline{\underline{\Gamma}}\left(\sum_{r=1}^{2}\underline{w}_{r}\left(\underline{p}\right)\underline{\bar{w}}_{r}\left(\underline{p}\right)\right)\underline{\underline{\beta}}\,\underline{\underline{\Gamma}}^{\dagger}\,\underline{\underline{\beta}}\,\underline{w}_{r'}\left(\underline{p}'\right)\\
&=\sum_{\alpha\beta\gamma\delta}\sum_{r',=1}^{2}\frac{1}{2}\underline{\bar{w}}_{r',\alpha}\left(\underline{p}'\right)\underline{\underline{\Gamma}}_{a\beta}\left(\sum_{r=1}^{2}\underline{w}_{r\beta}\left(\underline{p}\right)\underline{\bar{w}}_{r,\gamma}\left(\underline{p}\right)\right)\left(\underline{\underline{\beta}}\,\underline{\underline{\Gamma}}^{\dagger}\,\underline{\underline{\beta}}\right)_{\gamma\delta}\underline{w}_{r',\delta}\left(\underline{p}'\right),
\end{aligned}
\tag{6.2.29}
$$

where we now apply $(3.4.13)$ twice and introduce

$$
\underline{\underline{\bar{\Gamma}}}=\underline{\underline{\beta}}\,\underline{\underline{\Gamma}}^{\dagger}\,\underline{\underline{\beta}}\quad \text{like this} \tag{6.2.30}
$$

$$
\begin{aligned}
&\sum_{r.r'=1}^{2}\frac{1}{2}\left|\sum_{\mu,\nu=0}^{3} M_{fi,\mu\nu}\left(\underline{k}',\lambda',\underline{p}',r',\underline{k},\lambda,\underline{p},r\right)\right|^{2}\\
&=\sum_{\alpha\beta\gamma\delta}\sum_{r'=1}^{2}\frac{1}{2}\,\underline{\bar{w}}_{r',\alpha}\left(\underline{p}'\right)\underline{\underline{\Gamma}}_{a\beta}\left(\frac{c\underline{\underline{\not{p}}}+mc^{2}\underline{\underline{1}}}{2mc^{2}}\right)_{\beta\gamma}\underline{\underline{\bar{\Gamma}}}_{\gamma\delta}\underline{w}_{r',\delta}\left(\underline{p}'\right)\\
&=\sum_{\alpha\delta}\sum_{r'=1}^{2}\frac{1}{2}\left(\underline{\underline{\Gamma}}\frac{c\underline{\underline{\not{p}}}+mc^{2}\underline{\underline{1}}}{2mc^{2}}\underline{\underline{\bar{\Gamma}}}\right)_{\alpha\delta}\underline{w}_{r',\delta}\left(\underline{p}'\right)\underline{\bar{w}}_{r',\alpha}\left(\underline{p}'\right)\\
&=\sum_{\alpha\delta}\frac{1}{2}\left(\underline{\underline{\Gamma}}\frac{c\underline{\underline{\not{p}}}+mc^{2}\underline{\underline{1}}}{2mc^{2}}\underline{\underline{\bar{\Gamma}}}\right)_{\alpha\delta}\left(\frac{c\underline{\underline{\not{p}}}'+mc^{2}\underline{\underline{1}}}{2mc^{2}}\right)_{\delta\alpha}\\
&=\frac{1}{2}\mathrm{Tr}\left(\frac{c\underline{\underline{\not{p}}}'+mc^{2}\underline{\underline{1}}}{2mc^{2}}\underline{\underline{\Gamma}}\frac{c\underline{\underline{\not{p}}}+mc^{2}\underline{\underline{1}}}{2mc^{2}}\underline{\underline{\bar{\Gamma}}}\right).
\end{aligned}
\tag{6.2.31}
$$

In order to reduce $(6.2.31)$ we remodel Γ, $(6.2.27)$, using $(6.1.62)$ and the markings $\underline{\varepsilon}\equiv\underline{\varepsilon}_{\lambda}\left(\underline{k}\right)$ and $\underline{\varepsilon}'\equiv\underline{\varepsilon}_{\lambda'}\left(\underline{k}'\right)$

$$
\begin{aligned}
\underline{\underline{\Gamma}}=\hbar[&\frac{2\underline{p}\cdot\underline{\varepsilon}\;\underline{\underline{\not{\varepsilon}}}'+\underline{\underline{\not{\varepsilon}}}'\underline{\underline{\not{k}}}\,\underline{\underline{\not{\varepsilon}}}-\underline{\underline{\not{\varepsilon}}}'\underline{\underline{\not{\varepsilon}}}\left(\underline{\underline{\not{p}}}-\underline{\underline{1}}mc\right)}{2\underline{p}\cdot\underline{k}}\\
&+\frac{2\underline{p}\cdot\underline{\varepsilon}'\;\underline{\underline{\not{\varepsilon}}}-\underline{\underline{\not{\varepsilon}}}\,\underline{\underline{\not{k}}}'\underline{\underline{\not{\varepsilon}}}'-\underline{\underline{\not{\varepsilon}}}\,\underline{\underline{\not{\varepsilon}}}'\left(\underline{\underline{\not{p}}}-\underline{\underline{1}}mc\right)}{-2\underline{p}\cdot\underline{k}'}].
\end{aligned}
\tag{6.2.32}
$$

Using the mathematical relation $\left(\underline{\underline{A}}\,\underline{\underline{B}}\,\underline{\underline{C}}\right)^{\dagger}=\underline{\underline{C}}^{\dagger}\underline{\underline{B}}^{\dagger}\underline{\underline{A}}^{\dagger}$ one obtains

$$\underline{\underline{\bar{\Gamma}}} = \underline{\underline{\beta}}\underline{\underline{\Gamma}}^\dagger\underline{\underline{\beta}} = \hbar\underline{\underline{\beta}}[\frac{2\underline{p}\cdot\underline{\varepsilon}\,\underline{\underline{\varepsilon\!\!/}}'^\dagger + \underline{\underline{\varepsilon\!\!/}}^\dagger\underline{\underline{k\!\!/}}^\dagger\underline{\underline{\varepsilon\!\!/}}'^\dagger - \left(\underline{\underline{p\!\!/}} - \underline{\underline{1}}mc\right)^\dagger\underline{\underline{\varepsilon\!\!/}}^\dagger\underline{\underline{\varepsilon\!\!/}}'^\dagger}{2\underline{p}\cdot\underline{k}}$$

$$\frac{2\underline{p}\cdot\underline{\varepsilon}'\,\underline{\underline{\varepsilon\!\!/}}^\dagger - \underline{\underline{\varepsilon\!\!/}}'^\dagger\underline{\underline{k\!\!/}}'^\dagger\underline{\underline{\varepsilon\!\!/}}^\dagger - \left(\underline{\underline{p\!\!/}} - \underline{\underline{1}}mc\right)^\dagger\underline{\underline{\varepsilon\!\!/}}'^\dagger\underline{\underline{\varepsilon\!\!/}}^\dagger}{-2\underline{p}\cdot\underline{k}'}]\underline{\underline{\beta}}. \qquad (6.2.33)$$

Because of $\underline{\underline{\beta}}\underline{\underline{\varepsilon\!\!/}}^\dagger\underline{\underline{\beta}} = \underline{\underline{\beta}}\left(\sum_\mu \varepsilon^{(\mu)} g_{\mu\mu}\underline{\underline{\beta}}\underline{\underline{\alpha}}_\mu\right)^\dagger\underline{\underline{\beta}} = \underline{\underline{\beta}}\sum_\mu \varepsilon^{(\mu)} g_{\mu\mu}\underline{\underline{\alpha}}_\mu\underline{\underline{\beta}}\underline{\underline{\beta}} = \underline{\underline{\varepsilon\!\!/}},$

$$\underline{\underline{\beta}}\underline{\underline{k\!\!/}}^\dagger\underline{\underline{\beta}} = \underline{\underline{k\!\!/}} \text{ and } \underline{\underline{\beta}}\left(\underline{\underline{p}} - \underline{\underline{1}}mc\right)^\dagger\cdot\underline{\underline{\beta}} = \underline{\underline{p}} - \underline{\underline{1}}mc$$

we have $\underline{\underline{\bar{\Gamma}}} = \hbar[\frac{2\underline{p}\cdot\underline{\varepsilon}\,\underline{\underline{\varepsilon\!\!/}}' + \underline{\underline{\varepsilon\!\!/}}\underline{\underline{k\!\!/}}\underline{\underline{\varepsilon\!\!/}}' - \left(\underline{\underline{\bar{p\!\!/}}} - \underline{\underline{1}}mc\right)\underline{\underline{\varepsilon\!\!/}}\underline{\underline{\varepsilon\!\!/}}'}{2\underline{p}\cdot\underline{k}} + \frac{2\underline{p}\cdot\underline{\varepsilon}'\,\underline{\underline{\varepsilon\!\!/}} - \underline{\underline{\varepsilon\!\!/}}'\underline{\underline{k\!\!/}}'\underline{\underline{\varepsilon\!\!/}} - \left(\underline{\underline{\bar{p\!\!/}}} - \underline{\underline{1}}mc\right)\underline{\underline{\varepsilon\!\!/}}'\underline{\underline{\varepsilon\!\!/}}}{-2\underline{p}\cdot\underline{k}'}].$

The following product will appear in $(6.2.31)$

$$\left(\underline{\underline{p\!\!/}} - mc\underline{\underline{1}}\right)\left(\underline{\underline{p\!\!/}} + mc\underline{\underline{1}}\right) = \underline{\underline{p\!\!/}}^2 - m^2c^2\underline{\underline{1}}.$$

Using $(3.4.12\ )$and $(6.2.8)$we can write in the laboratory reference frame

$$\left(\underline{\underline{p\!\!/}} - mc\underline{\underline{1}}\right)\left(\underline{\underline{p\!\!/}} + mc\underline{\underline{1}}\right) = \underline{\underline{1}}\left((mc)^2 - 0 - (mc)^2\right) = 0 \qquad (6.2.34)$$

We insert $(6.2.32)$up to $(6.2.34)$ in $(6.2.31)$ like this

$$\sum_{r',r}\frac{1}{2}\left|\sum_{\mu,\nu} M_{fi,\mu\nu}\right|^2$$

$$= \hbar^2\frac{1}{2}\mathrm{Tr}[\frac{c\underline{\underline{p\!\!/}}' + mc^2\underline{\underline{1}}}{2mc^2}\left(\frac{2\underline{p}\cdot\underline{\varepsilon}\,\underline{\underline{\varepsilon\!\!/}}' + \underline{\underline{\varepsilon\!\!/}}'\underline{\underline{k\!\!/}}\underline{\underline{\varepsilon\!\!/}}}{2\underline{p}\cdot\underline{k}} + \frac{2\underline{p}\cdot\underline{\varepsilon}'\,\underline{\underline{\varepsilon\!\!/}} - \underline{\underline{\varepsilon\!\!/}}\underline{\underline{k\!\!/}}'\underline{\underline{\varepsilon\!\!/}}'}{-2\underline{p}\cdot\underline{k}'}\right)$$

$$\cdot\frac{c\underline{\underline{p\!\!/}} + mc^2\underline{\underline{1}}}{2mc^2}\underline{\underline{\bar{\Gamma}}}] \qquad (6.2.35)$$

$$= \left(\frac{\hbar}{4mc^2}\right)^2\frac{1}{2}\mathrm{Tr}[\left(c\underline{\underline{p\!\!/}}' + mc^2\underline{\underline{1}}\right)\left(\frac{2\underline{p}\cdot\underline{\varepsilon}\,\underline{\underline{\varepsilon\!\!/}}' + \underline{\underline{\varepsilon\!\!/}}'\underline{\underline{k\!\!/}}\underline{\underline{\varepsilon\!\!/}}}{\underline{p}\cdot\underline{k}} - \frac{2\underline{p}\cdot\underline{\varepsilon}'\,\underline{\underline{\varepsilon\!\!/}} - \underline{\underline{\varepsilon\!\!/}}\underline{\underline{k\!\!/}}'\underline{\underline{\varepsilon\!\!/}}'}{\underline{p}\cdot\underline{k}'}\right)$$

$$\cdot\left(c\underline{\underline{p\!\!/}} + mc^2\underline{\underline{1}}\right)\left(\frac{2\underline{p}\cdot\underline{\varepsilon}\,\underline{\underline{\varepsilon\!\!/}}' + \underline{\underline{\varepsilon\!\!/}}\underline{\underline{k\!\!/}}\underline{\underline{\varepsilon\!\!/}}'}{\underline{p}\cdot\underline{k}} - \frac{2\underline{p}\cdot\underline{\varepsilon}'\,\underline{\underline{\varepsilon\!\!/}} - \underline{\underline{\varepsilon\!\!/}}'\underline{\underline{k\!\!/}}'\underline{\underline{\varepsilon\!\!/}}}{\underline{p}\cdot\underline{k}'}\right)].$$

In electrodynamics one can perform a four dimensional gauge transformation with the polarization vectors $\underline{\varepsilon}$ and $\underline{\varepsilon}'$ (Greiner, Reinhardt, 1994, p.195) as follows

$$\underline{\tilde{\varepsilon}} = \underline{\varepsilon} - \frac{\underline{p}\cdot\underline{\varepsilon}}{\underline{p}\cdot\underline{k}}\underline{k}, \qquad \underline{\tilde{\varepsilon}}' = \underline{\varepsilon}' - \frac{\underline{p}\cdot\underline{\varepsilon}'}{\underline{p}\cdot\underline{k}'}\underline{k}', \tag{6.2.36}$$

which yields (using $(4.4.6)$ up to $(4.4.10)$, $(6.2.8)$ and $(6.2.12)$)

$$\underline{\tilde{\varepsilon}}_{\lambda=0}\left(\vec{k}\right) = \left(0, -\frac{\vec{k}}{\left|\vec{k}\right|}\right), \qquad \underline{\tilde{\varepsilon}}_{\lambda>0}\left(\vec{k}\right) = \left(0, \vec{\varepsilon}_{\lambda>0}\left(\vec{k}\right)\right), \tag{6.2.37}$$

$$\begin{aligned}
&\underline{p}\cdot\underline{\tilde{\varepsilon}} = 0, \qquad \underline{p}\cdot\underline{\tilde{\varepsilon}}' = 0, \qquad \underline{\tilde{\varepsilon}}\cdot\underline{\tilde{\varepsilon}} = -1, \qquad \underline{\tilde{\varepsilon}}'\cdot\underline{\tilde{\varepsilon}}' = -1, \\
&\text{for } \lambda = 0: \ \underline{\tilde{\varepsilon}}\cdot\underline{k} = \frac{\left|\vec{k}\right|^2}{\left|\vec{k}\right|} = \left|\vec{k}\right| \\
&\text{for } \lambda = 1, 2: \ \underline{\tilde{\varepsilon}}\cdot\underline{k} = 0 \\
&\text{for } \lambda = 3: \quad \underline{\tilde{\varepsilon}}\cdot\underline{k} = -\left|\vec{k}\right|.
\end{aligned} \tag{6.2.38}$$

Moreover, due to $(6.2.12)$ and $(6.1.62)$ we have

$$\begin{aligned}
&\underline{\underline{\not{k}}}^2 = \underline{k}^2\,\underline{\underline{1}} = 0, \qquad \underline{\underline{\not{k}}}'^2 = \underline{k}'^2\,\underline{\underline{1}} = 0, \\
&\underline{\underline{\tilde{\not{\varepsilon}}}}^2 = \underline{\tilde{\varepsilon}}^2\,\underline{\underline{1}} = -\underline{\underline{1}}, \qquad \underline{\underline{\tilde{\not{\varepsilon}}}}'^2 = -\underline{\underline{1}} \\
&\text{and for } \lambda = 1 \text{ and } 2: \quad \underline{\underline{\tilde{\not{\varepsilon}}}}\,\underline{\underline{\not{k}}} = -\underline{\underline{\not{k}}}\,\underline{\underline{\tilde{\not{\varepsilon}}}}.
\end{aligned} \tag{6.2.39}$$

Replacing $\underline{\varepsilon}$ and $\underline{\varepsilon}'$ in $(6.2.35)$ by $\underline{\tilde{\varepsilon}}$ and $\underline{\tilde{\varepsilon}}'$ and inserting $(6.2.38)$ we obtain

$$\begin{aligned}
\sum_{r',r}\frac{1}{2}\left|\sum_{\mu,\nu} M_{fi,\mu\nu}\right|^2 &= \left(\frac{\hbar}{4mc^2}\right)^2 \frac{1}{2} \\
\cdot\,\mathrm{Tr}\Big(\left(c\underline{\underline{\not{p}}}' + mc^2\,\underline{\underline{1}}\right)\Big[&\frac{\underline{\underline{\tilde{\not{\varepsilon}}}}'\,\underline{\underline{\not{k}}}\,\underline{\underline{\tilde{\not{\varepsilon}}}}\left(c\underline{\underline{\not{p}}} + mc^2\,\underline{\underline{1}}\right)\underline{\underline{\tilde{\not{\varepsilon}}}}\,\underline{\underline{\not{k}}}\,\underline{\underline{\tilde{\not{\varepsilon}}}}'}{\left(\underline{p}\cdot\underline{k}\right)^2} \\
&+ \frac{\underline{\underline{\tilde{\not{\varepsilon}}}}\,\underline{\underline{\not{k}}}'\,\underline{\underline{\tilde{\not{\varepsilon}}}}'\left(c\underline{\underline{\not{p}}} + mc^2\,\underline{\underline{1}}\right)\underline{\underline{\tilde{\not{\varepsilon}}}}'\,\underline{\underline{\not{k}}}'\,\underline{\underline{\tilde{\not{\varepsilon}}}}}{\left(\underline{p}\cdot\underline{k}'\right)^2} \\
&+ \frac{\underline{\underline{\tilde{\not{\varepsilon}}}}'\,\underline{\underline{\not{k}}}\,\underline{\underline{\tilde{\not{\varepsilon}}}}\left(c\underline{\underline{\not{p}}} + mc^2\,\underline{\underline{1}}\right)\underline{\underline{\tilde{\not{\varepsilon}}}}'\,\underline{\underline{\not{k}}}'\,\underline{\underline{\tilde{\not{\varepsilon}}}}}{\left(\underline{p}\cdot\underline{k}\right)\left(\underline{p}\cdot\underline{k}'\right)} \\
&+ \frac{\underline{\underline{\tilde{\not{\varepsilon}}}}\,\underline{\underline{\not{k}}}'\,\underline{\underline{\tilde{\not{\varepsilon}}}}'\left(c\underline{\underline{\not{p}}} + mc^2\,\underline{\underline{1}}\right)\underline{\underline{\tilde{\not{\varepsilon}}}}\,\underline{\underline{\not{k}}}\,\underline{\underline{\tilde{\not{\varepsilon}}}}'}{\left(\underline{p}\cdot\underline{k}\right)\left(\underline{p}\cdot\underline{k}'\right)}\Big]\Big)
\end{aligned} \tag{6.2.40}$$

We show that the last two expressions in the square brackets of $(6.2.40)$ are identical. If there is a trace of a product of an even number of Feynman daggers, the sequence of these daggers can be reversed like this

$$\mathrm{Tr}\left(\underline{\underline{\not{a}}}_1\underline{\underline{\not{a}}}_2\cdots\underline{\underline{\not{a}}}_{2n}\right)=\mathrm{Tr}\left(\underline{\underline{\not{a}}}_{2n}\cdots\underline{\underline{\not{a}}}_2\underline{\underline{\not{a}}}_1\right), \tag{6.2.41}$$

which is proven by Greiner, Reinhardt, 1994, p. 99 and 363. We write the nominator of the last but one expression in $(6.2.40)$ omitting mixed terms (with one factor mc^2, containing an odd number of Feynman daggers, (c.f. $(6.1.66\mathrm{a})$) and following $(6.2.41)$ like this

$$\begin{aligned}
&\mathrm{Tr}\left(\left(c\underline{\underline{\not{p}}}'+mc^2\underline{\underline{1}}\right)\underline{\underline{\tilde{\not{\varepsilon}}}}'\underline{\underline{\not{k}}}\,\underline{\underline{\tilde{\not{\varepsilon}}}}\left(c\underline{\underline{\not{p}}}+mc^2\underline{\underline{1}}\right)\underline{\underline{\tilde{\not{\varepsilon}}}}'\underline{\underline{\not{k}}}'\underline{\underline{\tilde{\not{\varepsilon}}}}\right)\\
&=\mathrm{Tr}\left(c\underline{\underline{\not{p}}}'\underline{\underline{\tilde{\not{\varepsilon}}}}'\underline{\underline{\not{k}}}\,\underline{\underline{\tilde{\not{\varepsilon}}}}c\underline{\underline{\not{p}}}\,\underline{\underline{\tilde{\not{\varepsilon}}}}'\underline{\underline{\not{k}}}'\underline{\underline{\tilde{\not{\varepsilon}}}}\right)+\mathrm{Tr}\left(mc^2\underline{\underline{1}}\,\underline{\underline{\tilde{\not{\varepsilon}}}}'\underline{\underline{\not{k}}}\,\underline{\underline{\tilde{\not{\varepsilon}}}}mc^2\underline{\underline{1}}\,\underline{\underline{\tilde{\not{\varepsilon}}}}'\underline{\underline{\not{k}}}'\underline{\underline{\tilde{\not{\varepsilon}}}}\right)\\
&=\mathrm{Tr}\left(\underline{\underline{\tilde{\not{\varepsilon}}}}\underline{\underline{\not{k}}}'\underline{\underline{\tilde{\not{\varepsilon}}}}'c\underline{\underline{\not{p}}}\,\underline{\underline{\tilde{\not{\varepsilon}}}}\underline{\underline{\not{k}}}\,\underline{\underline{\tilde{\not{\varepsilon}}}}'c\underline{\underline{\not{p}}}'\right)+\mathrm{Tr}\left(\underline{\underline{\tilde{\not{\varepsilon}}}}\underline{\underline{\not{k}}}'\underline{\underline{\tilde{\not{\varepsilon}}}}'mc^2\underline{\underline{1}}\,\underline{\underline{\tilde{\not{\varepsilon}}}}\underline{\underline{\not{k}}}\,\underline{\underline{\tilde{\not{\varepsilon}}}}'mc^2\underline{\underline{1}}\right)\\
&=\mathrm{Tr}\left(\underline{\underline{\tilde{\not{\varepsilon}}}}\underline{\underline{\not{k}}}'\underline{\underline{\tilde{\not{\varepsilon}}}}'\left(c\underline{\underline{\not{p}}}+mc^2\underline{\underline{1}}\right)\underline{\underline{\tilde{\not{\varepsilon}}}}\underline{\underline{\not{k}}}\,\underline{\underline{\tilde{\not{\varepsilon}}}}'\left(c\underline{\underline{\not{p}}}'+mc^2\underline{\underline{1}}\right)\right)\\
&=\mathrm{Tr}\left(\left(c\underline{\underline{\not{p}}}'+mc^2\underline{\underline{1}}\right)\underline{\underline{\tilde{\not{\varepsilon}}}}\underline{\underline{\not{k}}}'\underline{\underline{\tilde{\not{\varepsilon}}}}'\left(c\underline{\underline{\not{p}}}+mc^2\underline{\underline{1}}\right)\underline{\underline{\tilde{\not{\varepsilon}}}}\underline{\underline{\not{k}}}\,\underline{\underline{\tilde{\not{\varepsilon}}}}'\right).
\end{aligned} \tag{6.2.42}$$

The last line of $(6.2.42)$ follows from $(6.1.55)$. Thus, the last two lines of $(6.2.40)$ are identical. Therefore, $(6.2.40)$ reads now

$$\sum_{r',r}\frac{1}{2}\left|\sum_{\mu,\nu}M_{fi,\mu\nu}\right|^2=\left(\frac{\hbar}{4mc^2}\right)^2\frac{1}{2}\left(\frac{S_1}{\left(\underline{p}\cdot\underline{k}\right)^2}+\frac{S_2}{\left(\underline{p}\cdot\underline{k}'\right)^2}+\frac{2S_3}{\left(\underline{p}\cdot\underline{k}\right)\left(\underline{p}\cdot\underline{k}'\right)}\right) \tag{6.2.43}$$

First we deal with the trace S_1

$$\begin{aligned}
S_1&=\mathrm{Tr}\left(\left(c\underline{\underline{\not{p}}}'+mc^2\underline{\underline{1}}\right)\underline{\underline{\tilde{\not{\varepsilon}}}}'\underline{\underline{\not{k}}}\,\underline{\underline{\tilde{\not{\varepsilon}}}}\left(c\underline{\underline{\not{p}}}+mc^2\underline{\underline{1}}\right)\underline{\underline{\tilde{\not{\varepsilon}}}}\underline{\underline{\not{k}}}\,\underline{\underline{\tilde{\not{\varepsilon}}}}'\right)\\
&=\mathrm{Tr}\left(\left(c\underline{\underline{\not{p}}}'+mc^2\underline{\underline{1}}\right)\underline{\underline{\tilde{\not{\varepsilon}}}}'\underline{\underline{\not{k}}}\,\underline{\underline{\tilde{\not{\varepsilon}}}}c\underline{\underline{\not{p}}}\,\underline{\underline{\tilde{\not{\varepsilon}}}}\underline{\underline{\not{k}}}\,\underline{\underline{\tilde{\not{\varepsilon}}}}'\right)\\
&\quad+\mathrm{Tr}\left(\left(c\underline{\underline{\not{p}}}'+mc^2\underline{\underline{1}}\right)\underline{\underline{\tilde{\not{\varepsilon}}}}'\underline{\underline{\not{k}}}\,\underline{\underline{\tilde{\not{\varepsilon}}}}mc^2\underline{\underline{1}}\,\underline{\underline{\tilde{\not{\varepsilon}}}}\underline{\underline{\not{k}}}\,\underline{\underline{\tilde{\not{\varepsilon}}}}'\right)
\end{aligned} \tag{6.2.44}$$

Due to $(6.2.39)$ the last term in $(6.2.44)$ vanishes and using $(6.1.62)$ and $(6.2.38)$ we can write

$$\begin{aligned}
S_1&=\mathrm{Tr}\left(\left(c\underline{\underline{\not{p}}}'+mc^2\underline{\underline{1}}\right)\underline{\underline{\tilde{\not{\varepsilon}}}}'\underline{\underline{\not{k}}}\,\underline{\underline{\tilde{\not{\varepsilon}}}}\left(-\underline{\underline{\tilde{\varepsilon}}}c\underline{\underline{\not{p}}}\right)\underline{\underline{\not{k}}}\,\underline{\underline{\tilde{\not{\varepsilon}}}}'\right)\\
&=c\mathrm{Tr}\left(\left(c\underline{\underline{\not{p}}}'+mc^2\underline{\underline{1}}\right)\underline{\underline{\tilde{\not{\varepsilon}}}}'\underline{\underline{\not{k}}}\underline{\underline{\not{p}}}\underline{\underline{\not{k}}}\,\underline{\underline{\tilde{\not{\varepsilon}}}}'\right)\\
&=c\mathrm{Tr}\left(\left(c\underline{\underline{\not{p}}}'+mc^2\underline{\underline{1}}\right)\underline{\underline{\tilde{\not{\varepsilon}}}}'\underline{\underline{\not{k}}}2\underline{p}\cdot\underline{k}\,\underline{\underline{\tilde{\not{\varepsilon}}}}'\right)-c\mathrm{Tr}\left(\left(c\underline{\underline{\not{p}}}'+mc^2\underline{\underline{1}}\right)\underline{\underline{\tilde{\not{\varepsilon}}}}\underline{\underline{\not{k}}}\underline{\underline{\not{k}}}\underline{\underline{\not{p}}}\,\underline{\underline{\tilde{\not{\varepsilon}}}}'\right),
\end{aligned} \tag{6.2.45}$$

where the last expression vanishes again. Because of

$$\mathrm{Tr}\left(\underline{\underline{\tilde{\not{\varepsilon}}}}'\underline{\underline{\not{k}}}\,\underline{\underline{\tilde{\not{\varepsilon}}}}'\right)=-\mathrm{Tr}\left(\underline{\underline{\not{k}}}\right)=0,$$

(c.f. $(6.2.39)$) we obtain

$$\begin{aligned} S_1 &= c^2 2\underline{p}\cdot\underline{k}\ \mathrm{Tr}\left(\underline{\underline{\not p}}'\underline{\underline{\tilde{\not\varepsilon}}}'\underline{\underline{\not k}}\,\underline{\underline{\tilde{\not\varepsilon}}}'\right) \\ &= c^2 2\underline{p}\cdot\underline{k}\left(2\underline{k}\cdot\underline{\tilde{\varepsilon}}'\ \mathrm{Tr}\left(\underline{\underline{\not p}}'\underline{\underline{\tilde{\not\varepsilon}}}'\right)\text{-}\mathrm{Tr}\left(\underline{\underline{\not p}}'\underline{\underline{\tilde{\not\varepsilon}}}'\underline{\underline{\tilde{\not\varepsilon}}}'\underline{\underline{\not k}}\right)\right). \end{aligned} \tag{6.2.46}$$

With the help of $(6.1.65)$ S_1 becomes

$$S_1 = c^2 2\underline{p}\cdot\underline{k}\left(2\underline{k}\cdot\underline{\tilde{\varepsilon}}'\ 4\underline{p}'\cdot\underline{\tilde{\varepsilon}}' + 4\underline{p}'\cdot\underline{k}\right). \tag{6.2.47}$$

We go on simplifying S_1. Because of

$$\underline{p} = \left(mc,\vec{0}\right), \quad \underline{p}' = \left(\frac{E'}{c},\vec{p}'\right) = \left(\frac{\sqrt{c^2\vec{p}'^2 + m^2c^4}}{c},\vec{p}'\right)$$

the following relation holds

$$\underline{p}^2 = m^2c^2, \quad \underline{p}'^2 = \frac{c\vec{p}'^2 + m^2c^4}{c^2} - \vec{p}'^2 = m^2c^2. \tag{6.2.48}$$

From the δ-function in $(6.2.1)$ the following energy-momentum relation results

$$\underline{k}' + \underline{p}' = \underline{k} + \underline{p} \ \text{ or } \ \underline{p}' - \underline{k} = \underline{p} - \underline{k}', \tag{6.2.49}$$

which we square

$$\begin{aligned} \left(\underline{p}' - \underline{k}\right)^2 &= \left(\underline{p} - \underline{k}'\right)^2 \\ \underline{p}'^2 - 2\underline{p}'\cdot\underline{k} + \underline{k}^2 &= \underline{p}^2 - 2\underline{p}\cdot\underline{k}' + \underline{k}'^2. \end{aligned} \tag{6.2.50}$$

Due to $(6.2.39)$ and $(6.2.48)$ we obtain

$$\underline{p}'\cdot\underline{k} = \underline{p}\cdot\underline{k}'. \tag{6.2.51}$$

Restricting ourselves to "realistic" photon polarizations with λ=1 and 2 we state (c.f. $(6.2.36/37)$)

$$\underline{\tilde{\varepsilon}}'\cdot\underline{k}' = 0 - \vec{\varepsilon}'\cdot\vec{k}' = 0 \qquad \text{(c.f. } (4.4.7)\text{)}. \tag{6.2.52}$$

We multiply the energy-momentum relation $(6.2.49)$ by $\underline{\tilde{\varepsilon}}'$ and using $(6.2.38)$ and $(6.2.52)$ we obtain

$$\underline{\tilde{\varepsilon}}'\cdot\underline{p}' - \underline{\tilde{\varepsilon}}'\cdot\underline{k} = \underline{\tilde{\varepsilon}}'\cdot\underline{p} - \underline{\tilde{\varepsilon}}'\cdot\underline{k}' = 0 \tag{6.2.53}$$

We insert $(6.2.51)$ and $(6.2.53)$ in $(6.2.47)$ like this

$$S_1 = 8c^2\underline{p}\cdot\underline{k}\left(\underline{p}\cdot\underline{k}' + 2\left(\underline{k}\cdot\underline{\tilde{\varepsilon}}'\right)^2\right). \tag{6.2.54}$$

Now, we deal with the second trace in $(6.2.43)$

$$S_2 = \mathrm{Tr}\left(\left(c\underline{\underline{p\!\!\!/}}' + mc^2\underline{\underline{1}}\right)\underline{\underline{\tilde{\varepsilon\!\!/}}}\underline{\underline{K\!\!\!/}}'\underline{\underline{\tilde{\varepsilon\!\!/}}}'\left(c\underline{\underline{p\!\!\!/}} + mc^2\underline{\underline{1}}\right)\underline{\underline{\tilde{\varepsilon\!\!/}}}'\underline{\underline{K\!\!\!/}}'\underline{\underline{\tilde{\varepsilon\!\!/}}}\right). \tag{6.2.55}$$

In analogy to the derivation of S_1, $(6.2.45)$ up to $(6.2.54)$, one finds

$$S_2 = 8c^2\underline{p}\cdot\underline{k}'\left(\underline{p}\cdot\underline{k} - 2\left(\underline{k}'\cdot\underline{\tilde{\varepsilon}}\right)^2\right). \tag{6.2.56}$$

Finally we have to calculate the trace S_3 from $(6.2.43)$ using $(6.2.40)$ and $(6.2.42)$

$$S_3 = \mathrm{Tr}\left(\left(c\underline{\underline{p\!\!\!/}}' + mc^2\underline{\underline{1}}\right)\underline{\underline{\tilde{\varepsilon\!\!/}}}'\underline{\underline{K\!\!\!/}}\,\underline{\underline{\varepsilon\!\!/}}\left(c\underline{\underline{p\!\!\!/}} + mc^2\underline{\underline{1}}\right)\underline{\underline{\tilde{\varepsilon\!\!/}}}'\underline{\underline{K\!\!\!/}}'\underline{\underline{\varepsilon\!\!/}}\right), \tag{6.2.57}$$

where we replace $\underline{\underline{p\!\!\!/}}'$ using $(6.2.49)$ as follows

$$\begin{aligned} S_3 &= \mathrm{Tr}\left(\left(c\underline{\underline{p\!\!\!/}} + mc^2\underline{\underline{1}}\right)\underline{\underline{\tilde{\varepsilon\!\!/}}}'\underline{\underline{K\!\!\!/}}\,\underline{\underline{\tilde{\varepsilon\!\!/}}}\left(c\underline{\underline{p\!\!\!/}} + mc^2\underline{\underline{1}}\right)\underline{\underline{\tilde{\varepsilon\!\!/}}}'\underline{\underline{K\!\!\!/}}'\underline{\underline{\tilde{\varepsilon\!\!/}}}\right) \\ &\quad + \mathrm{Tr}\left(c\left(\underline{\underline{K\!\!\!/}} - \underline{\underline{K\!\!\!/}}'\right)\underline{\underline{\tilde{\varepsilon\!\!/}}}'\underline{\underline{K\!\!\!/}}\,\underline{\underline{\tilde{\varepsilon\!\!/}}}c\underline{\underline{p\!\!\!/}}\,\underline{\underline{\tilde{\varepsilon\!\!/}}}'\underline{\underline{K\!\!\!/}}'\underline{\underline{\tilde{\varepsilon\!\!/}}}\right) \\ &\equiv S_{3a} + S_{3b}. \end{aligned} \tag{6.2.58}$$

We have omitted $mc^2\underline{\underline{1}}$ in the last bracket of $(6.2.58)$ because the corresponding term contains an odd number of Feynman daggers (c.f.$(6.1.66a)$).

We repeatedly apply $(6.1.62)$ to a part of the argument of S_{3a} and use $(6.2.38)$ and $(6.2.39)$

$$\begin{aligned} &\left(c\underline{\underline{p\!\!\!/}} + mc^2\underline{\underline{1}}\right)\underline{\underline{\tilde{\varepsilon\!\!/}}}'\underline{\underline{\tilde{\varepsilon\!\!/}}}\underline{\underline{K\!\!\!/}}\left(c\underline{\underline{p\!\!\!/}} + mc^2\underline{\underline{1}}\right)\underline{\underline{K\!\!\!/}}'\underline{\underline{\tilde{\varepsilon\!\!/}}}'\underline{\underline{\tilde{\varepsilon\!\!/}}} \\ &= \underline{\underline{\tilde{\varepsilon\!\!/}}}'\underline{\underline{\tilde{\varepsilon\!\!/}}}\left(c\underline{\underline{p\!\!\!/}} + mc^2\underline{\underline{1}}\right)\underline{\underline{K\!\!\!/}}\left(c\underline{\underline{p\!\!\!/}} + mc^2\underline{\underline{1}}\right)\underline{\underline{K\!\!\!/}}'\underline{\underline{\tilde{\varepsilon\!\!/}}}'\underline{\underline{\tilde{\varepsilon\!\!/}}} \\ &= \underline{\underline{\tilde{\varepsilon\!\!/}}}'\underline{\underline{\tilde{\varepsilon\!\!/}}}\left(c2\underline{k}\cdot\underline{p}\underline{\underline{1}} - c\underline{\underline{K\!\!\!/}}\underline{\underline{p\!\!\!/}} + mc^2\underline{\underline{K\!\!\!/}}\right)\left(c\underline{\underline{p\!\!\!/}} + mc^2\underline{\underline{1}}\right)\underline{\underline{K\!\!\!/}}'\underline{\underline{\tilde{\varepsilon\!\!/}}}'\underline{\underline{\tilde{\varepsilon\!\!/}}} \\ &= \underline{\underline{\tilde{\varepsilon\!\!/}}}'\underline{\underline{\tilde{\varepsilon\!\!/}}}\left(c2\underline{k}\cdot\underline{p}\left(c\underline{\underline{p\!\!\!/}} + mc^2\underline{\underline{1}}\right) + \underline{\underline{K\!\!\!/}}\left(-c\underline{\underline{p\!\!\!/}} + mc^2\underline{\underline{1}}\right)\left(c\underline{\underline{p\!\!\!/}} + mc^2\underline{\underline{1}}\right)\right)\underline{\underline{K\!\!\!/}}'\underline{\underline{\tilde{\varepsilon\!\!/}}}'\underline{\underline{\tilde{\varepsilon\!\!/}}}. \end{aligned} \tag{6.2.59}$$

The last summand of $(6.2.59)$ vanishes according to $(6.2.34)$. We obtain

$$S_{3a} = c2\underline{p}\cdot\underline{k}\mathrm{Tr}\left(\underline{\underline{\tilde{\varepsilon\!\!/}}}'\underline{\underline{\tilde{\varepsilon\!\!/}}}c\underline{\underline{p\!\!\!/}}\underline{\underline{K\!\!\!/}}'\underline{\underline{\tilde{\varepsilon\!\!/}}}'\underline{\underline{\tilde{\varepsilon\!\!/}}}\right). \tag{6.2.60}$$

Here the term with mc^2 also has vanished because the corresponding trace contains an odd number of Feynman daggers. We go on handling S_{3a} using $(6.1.62)$

$$S_{3a} = c^2 2\underline{p}\cdot\underline{k}\left(2\underline{\tilde{\varepsilon}}\cdot\underline{\tilde{\varepsilon}}'\,\mathrm{Tr}\left(\underline{\underline{\not{p}}}\,\underline{\underline{\not{k}}}'\,\underline{\underline{\tilde{\not\varepsilon}}}'\underline{\underline{\tilde{\not\varepsilon}}}\right)-\mathrm{Tr}\left(\underline{\underline{\tilde{\not\varepsilon}}}\,\underline{\underline{\tilde{\not\varepsilon}}}'\,\underline{\underline{\not{p}}}\,\underline{\underline{\not{k}}}'\,\underline{\underline{\tilde{\not\varepsilon}}}'\underline{\underline{\tilde{\not\varepsilon}}}\right)\right)$$

In the last trace mathematics allow to put the last $\underline{\underline{\tilde{\not\varepsilon}}}$ at the beginning of the argument list. Using $(6.2.39)$ and repeating the procedure with $\underline{\underline{\tilde{\not\varepsilon}}}'$ yields

$$S_{3a} = c^2 2\underline{p}\cdot\underline{k}\left(2\underline{\tilde{\varepsilon}}\cdot\underline{\tilde{\varepsilon}}'\,\mathrm{Tr}\left(\underline{\underline{\not{p}}}\,\underline{\underline{\not{k}}}'\,\underline{\underline{\tilde{\not\varepsilon}}}'\underline{\underline{\tilde{\not\varepsilon}}}\right)-\mathrm{Tr}\left(\underline{\underline{\not{p}}}\,\underline{\underline{\not{k}}}'\right)\right) \tag{6.2.61}$$

The second term in $(6.2.58)$ can be simplified. With the aid of $(6.1.62)$, $(6.2.52)$ and $(6.2.39)$ we write

$$\begin{aligned}
S_{3b} &= c^2\mathrm{Tr}\left(\left(\underline{\underline{\not{k}}}-\underline{\underline{\not{k}}}'\right)\underline{\underline{\tilde{\not\varepsilon}}}'\underline{\underline{\tilde{\not\varepsilon}}}\,\underline{\underline{\not{k}}}\,\underline{\underline{\not{p}}}\,\underline{\underline{\not{k}}}'\,\underline{\underline{\tilde{\not\varepsilon}}}'\underline{\underline{\tilde{\not\varepsilon}}}\right)\\
&= c^2\mathrm{Tr}\left(\underline{\underline{\not{k}}}\,\underline{\underline{\not\varepsilon}}'\underline{\underline{\not\varepsilon}}\,\underline{\underline{\not{k}}}\,\underline{\underline{\not{p}}}\,\underline{\underline{\not{k}}}'\,\underline{\underline{\not\varepsilon}}'\underline{\underline{\not\varepsilon}}+\underline{\underline{\not\varepsilon}}'\,\underline{\underline{\not{k}}}'\,\underline{\underline{\not\varepsilon}}\,\underline{\underline{\not{k}}}\,\underline{\underline{\not{p}}}\,\underline{\underline{\not{k}}}'\,\underline{\underline{\not\varepsilon}}'\underline{\underline{\not\varepsilon}}\right)\\
&= c^2 2\underline{\tilde{\varepsilon}}'\cdot\underline{k}\,\mathrm{Tr}\left(\underline{\underline{\tilde{\not\varepsilon}}}\,\underline{\underline{\not{k}}}\,\underline{\underline{\not{p}}}\,\underline{\underline{\not{k}}}'\,\underline{\underline{\tilde{\not\varepsilon}}}'\underline{\underline{\tilde{\not\varepsilon}}}\right)-c^2\mathrm{Tr}\left(\underline{\underline{\tilde{\not\varepsilon}}}'\left(\underline{\underline{\not{k}}}-\underline{\underline{\not{k}}}'\right)\underline{\underline{\tilde{\not\varepsilon}}}\,\underline{\underline{\not{k}}}\,\underline{\underline{\not{p}}}\,\underline{\underline{\not{k}}}'\,\underline{\underline{\tilde{\not\varepsilon}}}'\underline{\underline{\tilde{\not\varepsilon}}}\right)\\
&= c^2 2\underline{\tilde{\varepsilon}}'\cdot\underline{k}\,\mathrm{Tr}\left(\underline{\underline{\tilde{\not\varepsilon}}}\,\underline{\underline{\tilde{\not\varepsilon}}}\,\underline{\underline{\not{k}}}\,\underline{\underline{\not{p}}}\,\underline{\underline{\not{k}}}'\,\underline{\underline{\tilde{\not\varepsilon}}}'\right)-c^2\mathrm{Tr}\left(\underline{\underline{\tilde{\not\varepsilon}}}'\left(-\underline{\underline{\tilde{\not\varepsilon}}}\,\underline{\underline{\not{k}}}-\underline{\underline{\not{k}}}'\,\underline{\underline{\tilde{\not\varepsilon}}}\right)\underline{\underline{\not{k}}}\,\underline{\underline{\not{p}}}\,\underline{\underline{\not{k}}}'\,\underline{\underline{\tilde{\not\varepsilon}}}'\underline{\underline{\tilde{\not\varepsilon}}}\right)\\
&= -c^2 2\underline{\tilde{\varepsilon}}'\cdot\underline{k}\,\mathrm{Tr}\left(\underline{\underline{\not{k}}}\,\underline{\underline{\not{p}}}\,\underline{\underline{\not{k}}}'\,\underline{\underline{\tilde{\not\varepsilon}}}'\right)+c^2\mathrm{Tr}\left(\underline{\underline{\tilde{\not\varepsilon}}}'\,\underline{\underline{\not{k}}}'\,\underline{\underline{\tilde{\not\varepsilon}}}\,\underline{\underline{\not{k}}}\,\underline{\underline{\not{p}}}\,\underline{\underline{\not{k}}}'\,\underline{\underline{\tilde{\not\varepsilon}}}'\underline{\underline{\tilde{\not\varepsilon}}}\right)\\
&= -c^2 2\underline{\tilde{\varepsilon}}'\cdot\underline{k}\,\mathrm{Tr}\left(\underline{\underline{\not{k}}}\,\underline{\underline{\not{p}}}\,\underline{\underline{\not{k}}}'\,\underline{\underline{\tilde{\not\varepsilon}}}'\right)+c^2\mathrm{Tr}\left(\underline{\underline{\not{k}}}'\,\underline{\underline{\tilde{\not\varepsilon}}}'\underline{\underline{\tilde{\not\varepsilon}}}\,\underline{\underline{\not{k}}}\,\underline{\underline{\not{p}}}\,\underline{\underline{\tilde{\not\varepsilon}}}'\,\underline{\underline{\not{k}}}'\,\underline{\underline{\tilde{\not\varepsilon}}}\right)\\
&= -c^2 2\underline{\tilde{\varepsilon}}'\cdot\underline{k}\,\mathrm{Tr}\left(\underline{\underline{\not{k}}}\,\underline{\underline{\not{p}}}\,\underline{\underline{\not{k}}}'\,\underline{\underline{\tilde{\not\varepsilon}}}'\right)+c^2 2\underline{k}'\cdot\underline{\tilde{\varepsilon}}\,\mathrm{Tr}\left(\underline{\underline{\not{k}}}'\,\underline{\underline{\tilde{\not\varepsilon}}}'\underline{\underline{\tilde{\not\varepsilon}}}\,\underline{\underline{\not{k}}}\,\underline{\underline{\not{p}}}\,\underline{\underline{\tilde{\not\varepsilon}}}'\right)\\
&\quad -c^2\mathrm{Tr}\left(\underline{\underline{\not{k}}}'\,\underline{\underline{\tilde{\not\varepsilon}}}'\underline{\underline{\tilde{\not\varepsilon}}}\,\underline{\underline{\not{k}}}\,\underline{\underline{\not{p}}}\,\underline{\underline{\tilde{\not\varepsilon}}}'\underline{\underline{\tilde{\not\varepsilon}}}\,\underline{\underline{\not{k}}}'\right)\\
&= -c^2 2\underline{\tilde{\varepsilon}}'\cdot\underline{k}\,\mathrm{Tr}\left(\underline{\underline{\not{k}}}\,\underline{\underline{\not{p}}}\,\underline{\underline{\not{k}}}'\,\underline{\underline{\tilde{\not\varepsilon}}}'\right)-c^2 2\underline{k}'\cdot\underline{\tilde{\varepsilon}}\,\mathrm{Tr}\left(\underline{\underline{\tilde{\not\varepsilon}}}'\,\underline{\underline{\not{k}}}'\,\underline{\underline{\tilde{\not\varepsilon}}}\,\underline{\underline{\not{k}}}\,\underline{\underline{\not{p}}}\,\underline{\underline{\tilde{\not\varepsilon}}}'\right)\\
&= -c^2 2\underline{\tilde{\varepsilon}}'\cdot\underline{k}\,\mathrm{Tr}\left(\underline{\underline{\not{k}}}\,\underline{\underline{\not{p}}}\,\underline{\underline{\not{k}}}'\,\underline{\underline{\tilde{\not\varepsilon}}}'\right)+c^2 2\underline{k}'\cdot\underline{\tilde{\varepsilon}}\,\mathrm{Tr}\left(\underline{\underline{\not{k}}}'\,\underline{\underline{\tilde{\not\varepsilon}}}\,\underline{\underline{\not{k}}}\,\underline{\underline{\not{p}}}\right).
\end{aligned} \tag{6.2.62}$$

With the help of $(6.1.65)$, $(6.1.66)$, $(6.2.38)$ and $(6.2.52)$ we simplify the quantity $S_3 = S_{3a}+S_{3b}$ like this

$$\begin{aligned}
S_3 &= c^2 8\underline{p}\cdot\underline{k}\,[2\underline{\tilde{\varepsilon}}\cdot\underline{\tilde{\varepsilon}}'\left(\left(\underline{p}\cdot\underline{k}'\right)\left(\underline{\tilde{\varepsilon}}'\cdot\underline{\tilde{\varepsilon}}\right)-\left(\underline{p}\cdot\underline{\tilde{\varepsilon}}'\right)\left(\underline{k}'\cdot\underline{\tilde{\varepsilon}}\right)+\left(\underline{p}\cdot\underline{\tilde{\varepsilon}}\right)\left(\underline{k}'\cdot\underline{\tilde{\varepsilon}}'\right)\right)\\
&\quad -\underline{p}\cdot\underline{k}']\\
&\quad -c^2 8\underline{\tilde{\varepsilon}}'\cdot\underline{k}\left(\left(\underline{k}\cdot\underline{p}\right)\left(\underline{k}'\cdot\underline{\tilde{\varepsilon}}'\right)-\left(\underline{k}\cdot\underline{k}'\right)\left(\underline{p}\cdot\underline{\tilde{\varepsilon}}'\right)+\left(\underline{k}\cdot\underline{\tilde{\varepsilon}}'\right)\left(\underline{p}\cdot\underline{k}'\right)\right)\\
&\quad +c^2\underline{\tilde{8}}\underline{\tilde{\varepsilon}}\cdot\underline{k}'\left(\left(\underline{k}'\cdot\underline{\tilde{\varepsilon}}\right)\left(\underline{k}\cdot\underline{p}\right)-\left(\underline{k}'\cdot\underline{k}\right)\left(\underline{\tilde{\varepsilon}}\cdot\underline{p}\right)+\left(\underline{k}'\cdot\underline{p}\right)\left(\underline{\tilde{\varepsilon}}\cdot\underline{k}\right)\right) \qquad (6.2.63)\\
&= c^2 8\,[\left(\underline{k}\cdot\underline{p}\right)\left(\underline{k}'\cdot\underline{p}\right)\left(2\left(\underline{\tilde{\varepsilon}}'\cdot\underline{\tilde{\varepsilon}}\right)^2-1\right)-\left(\underline{k}\cdot\underline{\tilde{\varepsilon}}'\right)^2\left(\underline{p}\cdot\underline{k}'\right)\\
&\quad +\left(\underline{k}'\cdot\underline{\tilde{\varepsilon}}\right)^2\left(\underline{p}\cdot\underline{k}\right)].
\end{aligned}$$

Finally we can construct the differential photon scattering cross section, $(6.2.21)$. We insert $S_1, (6.2.54), S_2, (6.2.56)$, and $S_3, (6.2.63)$, in $\sum_{r,r'} \frac{1}{2} \left| \sum_{\mu,\nu} M_{fi,\mu\nu} \right|^2, (6.2.43)$, and again put this expression in $(6.2.21)$

$$\overline{\frac{d\sigma}{d\Omega_{k'}}}_{\lambda'\lambda} = \frac{1}{(4\pi)^2} \frac{\omega_{k'}^2}{\omega_k^2 c^2} \left(\frac{\mathcal{A}'\mathcal{E}'}{\mathcal{A}\mathcal{E}} \right)^4 \frac{\hbar^2}{32 m^2 c^4} \frac{e^4}{\varepsilon_0^2 \hbar^2} 8c^2$$
$$\cdot [\frac{\underline{p} \cdot \underline{k}' + 2(\underline{k} \cdot \underline{\tilde{\varepsilon}}')^2}{\underline{p} \cdot \underline{k}} + \frac{\underline{p} \cdot \underline{k} - 2(\underline{k}' \cdot \underline{\tilde{\varepsilon}})^2}{\underline{p} \cdot \underline{k}'}$$
$$+ 2 \left(2(\underline{\tilde{\varepsilon}}' \cdot \underline{\tilde{\varepsilon}})^2 - 1 - \frac{(\underline{k} \cdot \underline{\tilde{\varepsilon}}')^2}{\underline{p} \cdot \underline{k}} + \frac{(\underline{k}' \cdot \underline{\tilde{\varepsilon}})^2}{\underline{p} \cdot \underline{k}'} \right)]_{\omega_{k'} = \frac{mc^2 \omega_k}{mc^2 + \omega_k (1 - \cos\Theta)}} \qquad (6.2.64)$$
$$= \frac{1}{(4\pi)^2} \frac{\omega_{k'}^2}{\omega_k^2} \left(\frac{\mathcal{A}'\mathcal{E}'}{\mathcal{A}\mathcal{E}} \right)^4 \frac{\hbar^2}{4 m^2 c^4} \frac{e^4}{\varepsilon_0^2 \hbar^2}$$
$$\cdot \left[\frac{\underline{p} \cdot \underline{k}'}{\underline{p} \cdot \underline{k}} + \frac{\underline{p} \cdot \underline{k}}{\underline{p} \cdot \underline{k}'} + 4(\underline{\tilde{\varepsilon}}' \cdot \underline{\tilde{\varepsilon}})^2 - 2 \right]_{\omega_{k'} = \frac{mc^2 \omega_k}{mc^2 + \omega_k (1 - \cos\Theta)}}$$

where four terms have cancelled each other in pairs. We see again that the differential cross section has the dimension of an area because the expression

$$\left(\frac{\mathcal{A}'\mathcal{E}'}{\mathcal{A}\mathcal{E}} \right)^4 \qquad (6.2.65)$$

is dimensionless. Due to $(6.2.8)$ and $(6.2.12)$ we write

$$\underline{k}' \cdot \underline{p} = \omega_{k'} m, \quad , \underline{k} \cdot \underline{p} = \omega_k m. \qquad (6.2.66)$$

This leads to the well-known Klein-Nishina formula which describes Compton scattering of photons, where we insert the fine-structure constant

$$\alpha = \frac{e^2}{4\pi\varepsilon_0 \hbar c} \simeq \frac{1}{137} \qquad \text{(c.f. } (6.1.82)\text{)}.$$

$$\overline{\frac{d\sigma}{d\Omega_{k'}}}_{\lambda'\lambda} = \left(\frac{\alpha\hbar}{2mc} \right)^2 \left(\frac{\omega_{k'}}{\omega_k^2} \right)^2 \left(\frac{\omega_{k'}}{\omega_k} + \frac{\omega_k}{\omega_{k'}} + 4(\underline{\tilde{\varepsilon}} \cdot \underline{\tilde{\varepsilon}}')^2 - 2 \right). \qquad (6.2.67)$$

here we have put the dimensionless factor $(6.2.65)$ equal to the identity. From now on we will write

$$\alpha \frac{\hbar}{mc} \equiv r_0 \simeq 2.5 \cdot 10^{-15}\,\mathrm{m}. \qquad (6.2.68)$$

We now develop the unpolarized cross section of the Compton scattering i.e. we will admit all polarizations of the incoming photons $(\lambda = 1 \text{ and } 2)$ and take into account all polarizations of the scattered photons $(\lambda' = 1,\ 2)$. Thus, we shall sum over the polarizations λ' and average over the initial polarizations λ

$$\overline{\overline{\frac{d\sigma}{\Omega_{k'}}}} = \frac{1}{2}\sum_{\lambda,\lambda'=1}^{2}\overline{\frac{d\sigma}{\Omega_{k'}}}_{\lambda'\lambda}\,. \qquad (6.2.69)$$

The summation will concern specially the term $\left(\underline{\tilde{\varepsilon}}\cdot\underline{\tilde{\varepsilon}}'\right)^2$ in $(6.2.67)$. Due to $(6.2.37)$ we have

$$\underline{\tilde{\varepsilon}}_{\lambda}\left(\vec{k}\right) = \left(0, \vec{\varepsilon}_{\lambda}\left(\vec{k}\right)\right) \quad \lambda = 1,\ 2$$

and therefore

$$\underline{\tilde{\varepsilon}}\cdot\underline{\tilde{\varepsilon}}' = 0 - \vec{\varepsilon}_{\lambda}\left(\vec{k}\right)\cdot\vec{\varepsilon}_{\lambda'}\left(\vec{k}'\right). \qquad (6.2.70)$$

The spatial vectors $\vec{\varepsilon}_{\lambda=1}\left(\vec{k}\right), \vec{\varepsilon}_{\lambda=2}\left(\vec{k}\right)$ and $\vec{\mathrm{k}}$ form an orthogonal system, the same holds for the primed quantities. Now, without restricting generality we can choose $\vec{\varepsilon}_{\lambda=1}\left(\vec{k}\right)$ and $\varepsilon'_{\lambda'=1}\left(\vec{k}'\right)$ to lie in the plane spanned by $\vec{k}$ and $\vec{k}'$, see figure 6.2.1.

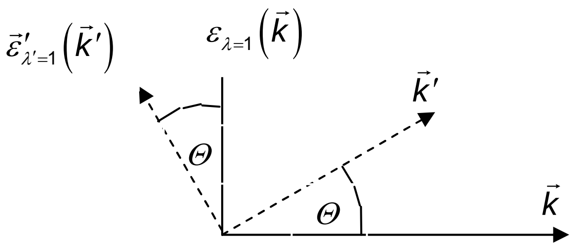

Fig. 6.2.1
Choice of the polarization vectors $\vec{\varepsilon}_{\lambda=1}\left(\vec{k}\right)$ and $\vec{\varepsilon}'_{\lambda'=1}\left(\vec{k}'\right)$ with respect to the plane of the propagation vectors $\vec{k}$ and $\vec{k}'$.

The angle between the vectors of polarization $\vec{\varepsilon}_{\lambda=1}\left(\vec{k}\right)$ and $\vec{\varepsilon}'_{\lambda'=1}\left(\vec{k}'\right), \Theta,$ is the same as the one between $\vec{k}$ and $\vec{k}'$. Then, $\vec{\varepsilon}_{\lambda=2}\left(\vec{k}\right)$ and $\vec{\varepsilon}'_{\lambda'=2}\left(\vec{k}'\right)$ are perpendicular to the plane $\left(\vec{k},\vec{k}'\right)$. We have the following spatial scalar products

$$\begin{aligned}
&\vec{\varepsilon}_{\lambda=1}\left(\vec{k}\right)\cdot\vec{\varepsilon}'_{\lambda'=1}\left(\vec{k}'\right) = \cos\Theta \\
&\vec{\varepsilon}_{\lambda=2}\left(\vec{k}\right)\cdot\vec{\varepsilon}'_{\lambda'=2}\left(\vec{k}'\right) = 1 \qquad (6.2.71)\\
&\vec{\varepsilon}_{\lambda=1}\left(\vec{k}\right)\cdot\vec{\varepsilon}'_{\lambda'=2}\left(\vec{k}'\right) = \vec{\varepsilon}_{\lambda=2}\left(\vec{k}\right)\cdot\vec{\varepsilon}'_{\lambda'=1}\left(\vec{k}'\right) = 0.
\end{aligned}$$

The averaged polarization dependent term in $(6.2.67)$ becomes

$$\frac{1}{2}\sum_{\lambda,\lambda'=1}^{2}\left(\vec{\varepsilon}_{\lambda}\left(\vec{k}\right)\cdot\vec{\varepsilon}'_{\lambda'}\left(\vec{k}'\right)\right)^2 = \frac{1}{2}\left(\cos^2\Theta + 1\right). \qquad (6.2.72)$$

Using this result and $(6.2.67)$ the unpolarized cross section for Compton scattering becomes

$$\overline{\overline{\frac{d\sigma}{d\Omega_{k'}}}} = \frac{r_0^2}{4}\frac{1}{2}4\frac{\omega_{k'}^2}{\omega_k^2}\left(\frac{\omega_{k'}}{\omega_k}+\frac{\omega_k}{\omega_{k'}}-2\right)+\frac{r_0^2}{4}\frac{1}{2}4\frac{\omega_{k'}^2}{\omega_k^2}\left(\cos^2\Theta+1\right)$$
$$= \frac{r_0^2}{2}\frac{\omega_{k'}^2}{\omega_k^2}\left(\frac{\omega_{k'}}{\omega_k}+\frac{\omega_k}{\omega_{k'}}-\sin^2\Theta\right). \qquad (6.2.73)$$

The classical limit of this result ($\omega_k \to 0$ and by means of $(6.2.18)$: $\frac{\omega_{k'}}{\omega_k} \to 1$) is the unpolarized Thomson cross section

$$\overline{\overline{\left(\frac{d\sigma}{d\Omega_{k'}}\right)_{\text{class}}}} = \frac{r_0^2}{2}\left(1+\cos^2\Theta\right). \qquad (6.2.74)$$

Figure 6.2.2 shows the result $(6.2.73)$ containing the limiting case $(6.2.74)$ and angular distributions for three different photon energies. At high energies the angular distribution gets concentrated in a narrow cone in the forward direction.

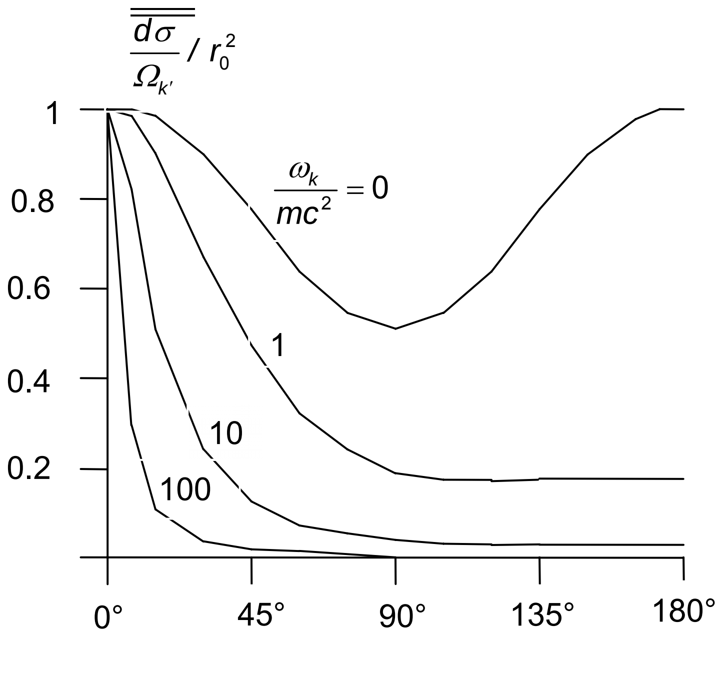

Figure 6.2.2
The differential cross section of unpolarized Compton scattering as a function of the scattering angle Θ for various photon energies ω_k.

7 Epilogue

We remind that this book is only introductory and not comprehensive. It intends to bring the newcomer as directly as possible in contact with the methods of quantum field theory. Therefore we have restricted ourselves to electron-electron and photon-electron scattering. In fact, these processes have already been described by quantum electrodynamics but they represent good introductory examples for quantum field theory and they might pave the way for other processes. We have abstained from the application of path integrals not to go beyond the scope of this introduction. Probably it is the first treatment of quantum field theory using the SI system of units, which is familiar to most beginners. We did not introduce the system of natural units because it offers little dimension controls.

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